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A TREATISE

ON

CONIC SECTIONS

AND THE

Wheelie

APPLICATION OF ALGEBRA TO GEOMETRY.

By J. HYMERS, M.A.

FELLOW AND TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

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CONIC SECTIONS

AND THE

APPLICATION OF ALGEBRA TO GEOMETRY.

SECTION I.

ON THE METHODS OF DETERMINING THE POSITION OF A POINT IN A PLANE.

Rectangular and oblique co-ordinates. Polar co-ordinates.

1. In order to determine the position of a point in a plane, some fixed point in the plane is taken for the origin of co-ordinates; and through it are drawn two fixed lines, called the co-ordinate axes, at right angles to one another.

Then if the perpendicular distances of a point from each of the co-ordinate axes be given, its position will be completely determined.

For let A (fig. 1), be the origin of co-ordinates, $X'AX$, $Y'AY$, the co-ordinate axes, P any point, and PM, PN the perpendiculars let fall from it upon the co-ordinate axes; these perpendiculars are called the rectangular co-ordinates of P, and as their values change for the different points of the plane, they are denoted by the variables x and y.

Then the point P will be determined in position, if we know the values of its two co-ordinates; that is, if we know that for that point $x = a$, $y = b$; for if along AX we

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measure $AN = a$, and through N draw an indefinite line parallel to AY, this line will contain all points in the plane whose distance from AY is a, or for which $x = a$, and therefore the point in question; similarly, if we measure along AY the distance $AM = b$, and through M draw an indefinite line parallel to AX, this line will contain the point in question; therefore these two lines MP, NP will by their intersection in P, determine one single position for the point whose co-ordinates are $x = a$, $y = b$; which position, as we see, coincides with the angular point opposite the origin of the rectangle constructed with the sides AN, AM equal to the two given co-ordinates.

2. Instead of the perpendicular PM, its equal AN is commonly used to determine the position of the point P; and the two AN, NP, are called the co-ordinates of P, and are denoted by x and y; the former, for the sake of distinction, being called the abscissa, as being cut off from AX, and the latter, which is parallel to the other axis AY, the ordinate.

When the point is given, and consequently its co-ordinates known, they are usually represented by the first letters of the alphabet a, b, &c. as above; or by the accented letters x', y', or x'', y''; also the axes of the co-ordinates AX, AY, are often called the axis of x and the axis of y.

3. The determination of the point P will not however be complete, unless we take into account the signs of the quantities a, b, in the equations

$$x = a, \quad y = b,$$

in order to measure these distances, when they are positive, along the positive parts AX, AY, of the co-ordinate axes; or along the negative parts AX', AY', of the axes produced in the contrary direction, when they are negative; as is explained in Trigonometry, (Art. 20). For since the co-ordinate axes, which must be supposed to be prolonged indefinitely, form about the origin four angular compartments, there are four positions in which P might be situated, at

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absolute distances, a, b, from the co-ordinate axes; and it is only attention to the algebraical signs, with which the values of those distances are affected, that will enable us to select the true one. The direction of the negative abscissæ is quite arbitrary, as is also that of the negative ordinates; we shall however, according to the usual practice, measure the positive abscissæ from the origin towards the right, and the negative abscissæ from the origin towards the left; and the positive ordinates we shall measure upwards from the axis of x, and the negative ordinates, downwards. Hence if the point P be situated in the compartment XAY, both its co-ordinates are positive; if in the opposite compartment $X'AY'$, both are negative; and for points in the compartments $X'AY$, XAY', we must have respectively

$$x = -a, \quad y = b; \quad x = a, \quad y = -b;$$

also for points in the axis of x, and axis of y, we shall have respectively

$$x = a, \quad y = 0; \quad x = 0, \quad y = b;$$

and for the origin, $x = 0$, $y = 0$.

4. Sometimes it is requisite to take the co-ordinate axes not at right angles, but inclined at a given angle to one another; in which case the system of co-ordinates is called oblique. Thus (fig. 2), if XAX', YAY', be two lines drawn through the point A, and intersecting one another at a given angle; and if from any point P in the plane XAY, PM, PN, be drawn respectively parallel to AX, AY, and meeting those axes in M and N; PM, or its equal AN, and NP are the co-ordinates of P, referred to the oblique axes AX, AY.

5. To find the distance of a point from the origin in terms of its co-ordinates.

Let P be the point (fig. 1).

$AN = x'$, $NP = y'$ its given co-ordinates. Join AP, and let $AP = d$; then from the triangle ANP, right-angled at N,

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$$AP^2 = AN^2 + NP^2, \text{ or } d^2 = x'^2 + y'^2,$$
$$\therefore \quad d = \sqrt{x'^2 + y'^2}.$$

6. To find the distance between two points in terms of their co-ordinates, and the angle of inclination of the line which joins them to the axis of x.

Let P' be a point (fig. 3), whose co-ordinates are x' and y'; and P any other point whose co-ordinates are x and y; join PP', and draw $P'Q$ parallel to AX meeting the ordinate of P in Q; then from the triangle PQP', right-angled at Q,

$$PP'^2 = P'Q^2 + PQ^2,$$
$$\text{or} \quad d^2 = (x - x')^2 + (y - y')^2,$$
$$\therefore \quad d = \sqrt{(x - x')^2 + (y - y')^2}.$$

Both in this formula and in that of Art. 5, we take the radical with a positive sign, as the question only relates to the absolute distance of the points.

Next let α be the angle which PP' forms with $P'Q$, and which is equal to the angle at which, if produced, PP' would be inclined to the axis of x,

$$\text{then} \quad \tan \alpha = \frac{PQ}{P'Q} = \frac{y - y'}{x - x'}.$$

7. Suppose the co-ordinates to be oblique, and the axes of the co-ordinates to be inclined to one another at an angle ω; then, for the distance of a point from the origin, we have (fig. 2)

$$AP^2 = AN^2 + NP^2 - 2\,AN \,.\, NP \cos ANP,$$
$$\text{but} \quad \cos ANP = -\cos XAY = -\cos \omega,$$
$$\therefore \quad d^2 = x^2 + y^2 + 2xy \cos \omega;$$

and for the distance between two points we have, in a similar manner, from the triangle PQP' (fig. 3), in which the angle $PQP' = \pi - \omega$,

$$d^2 = (x - x')^2 + (y - y')^2 + 2\,(x - x')\,(y - y') \cos \omega.$$

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8. There is also another mode of determining the position of a point in a plane, viz. by means of its distance from a given point or pole, and the angle which that distance makes with a fixed line or axis in the plane.

Let A (fig. 4), be the origin or pole, and AX a fixed line or axis; and P any point in the plane passing through AX. Join AP, then AP is called the radius vector, and is usually denoted by r, and the angle PAX is called the angle of revolution, and is denoted by θ; and r and θ are called the polar co-ordinates of P; and if given values $r = d$, $\theta = a$, be assigned for them, the position of P will be completely determined.

9. To express the distance of two points from one another in terms of their polar co-ordinates.

Let P' be a point (fig. 4) whose polar co-ordinates are r' and θ', and P any other point whose co-ordinates are r and θ; then the angle $PAP' = \theta - \theta'$, and joining PP', we get from the triangle PAP',

$$PP' \text{ or } d = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}.$$

Equation to a curve. Locus of an equation.

10. Knowing in this way how to determine the position of a point in a plane by means of its co-ordinates, conceive a curve line to be traced on a plane, and each of its points to be referred to two known axes; and that we have between the abscissa and ordinate of each point an invariable relation. In a great many cases it happens that this relation is of a nature to be expressed by an equation between the abscissa and ordinate; and this equation, when obtained, enables us to find either of these quantities by means of the other; so that giving to the abscissa, for instance, arbitrary values, we can deduce from the equation corresponding values of the ordinate; and we thus determine as many points of the curve as we please.

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The equation which expresses generally the invariable relation of the abscissa and ordinate of every point of a curve to one another, is called the equation to the curve: and, conversely, the curve is called the locus of the equation. Similarly, the equation which expresses the invariable relation of the radius vector and angle of revolution of a curve, is called the polar equation to the curve.

11. All lines are regular or irregular; irregular lines, described, as it is termed, liberâ manu, are not subjects of mathematical investigation, and cannot be represented by equations; but regular lines which are described according to some constant law, which determines the position of all their points, can be represented by equations. This idea of regular lines agrees with the geometrical loci of the ancients. They gave that name to those lines of which every point was equally proper to solve an indeterminate geometrical problem. Thus a circle was said to be the locus of the vertices of all triangles on a given base and having a given vertical angle. Des Cartes first adopted the method of expressing by an algebraical equation the nature of lines. The object of the following sections will be to investigate the equations to curves, and from those equations to discover their geometrical properties, by means of interpretations made according to the laws of Algebra.

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SECTION II.

ON THE STRAIGHT LINE.

Straight line referred to rectangular co-ordinates.

12. We will now suppose the locus of the point P to be a straight line, as defined in Geometry; and proceed to investigate its equation by means of some of its properties; that is, an equation expressing an invariable relation, which is satisfied by the co-ordinates of every point in the line.

13. To find the equation to a straight line.

Let A be the origin (fig. 5), AX the axis of x, AY that of y, RT the given straight line, P any point in it, and $AN = x$, $PN = y$ the co-ordinates of the point P.

Let $AB = c$, and the tangent of angle $PTN = m$. Draw BQ parallel to AX meeting PN in Q; then $PQ = BQ \cdot \tan PBQ = AN \cdot \tan PTN = mx$,

$$\text{and } PN = PQ + QN = PQ + AB,$$

$$\therefore y = mx + c;$$

and as this relation is satisfied by the co-ordinates of every point in the line, it is the equation required.

14. The meanings of the constants m and c are to be particularly observed; c is the part of the axis of y intercepted between the straight line and the origin, or the ordinate through the origin; m is the tangent of the angle which that part of the line which falls above the axis of x makes with the axis of x produced in the positive direction. They remain the same for the same line, but are different

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for different lines, and are called arbitrary constants; every straight line has two of them, and therefore a straight line may be drawn fulfilling two conditions.

The equation $y = mx + c$, which is the most convenient form and the one commonly employed, represents a straight line when determined by the conditions of passing through a known point in the axis of y, and making a given angle with a fixed line, viz. the axis of x; so that m is a number or ratio, denoting the tangent of the angle; and c denotes a line, viz. the distance from the origin of the point in the axis of y through which the line passes.

15. The equation to a straight line may also be put under the two following forms, which are sometimes useful.

Let $AT = -a$, the negative sign being prefixed because AT is measured towards the left from A; then

$$m = \tan BTA = -\frac{c}{a};$$

$$\therefore\ y = -\frac{c}{a}x + c,$$

$$\text{or } \frac{x}{a} + \frac{y}{c} = 1,$$

the equation to a straight line when determined by means of the portions of the co-ordinate axes intercepted between it and the origin.

Also if a perpendicular upon the straight line from the origin, $AD = p$, and the angle which the perpendicular forms with the axis of x produced in the positive direction, $DAX = \alpha$, we have $\tan BTA = -\cot\alpha$ and $c = \frac{p}{\sin\alpha}$, therefore

$$y = -x\cot\alpha + \frac{p}{\sin\alpha};\ \text{or } y\sin\alpha + x\cos\alpha = p,$$

the equation to a straight line when determined by the perpendicular upon it from the origin.

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16. The indeterminate equation of the first degree between two variables, is in its most general form

$$Ax + By + C = 0,$$

which in all cases is the equation to a straight line. For by putting

$$\frac{A}{B} = -m, \quad \frac{C}{B} = -c,$$

we reduce it to the form

$$y - mx - c = 0, \text{ or } y = mx + c,$$

which coincides with the equation to a straight line.

17. A straight line may always be determined from its equation

$$y = mx + c,$$

when the constants m and c are known.

First consider the equation $y = mx$, which represents a line passing through the origin; assume for x any positive value $AN = x'$ (fig. 6), take for y the value $NP = mx'$ (supposing m a positive quantity) and join AP, this is the required line. But if m be negative, so that the equation is $y = -mx$, taking $AN = x'$ and NP measured downwards $= mx'$ (fig 7), and joining AP, we have the line required.

We can now readily construct any line whatever whose equation is

$$y = mx + c.$$

For if in this equation, we give to x the same values that we assigned to it in the equation $y = mx$, the difference of the corresponding values of y will be constant and equal to c; the straight line which is the locus of $y = mx + c$ is consequently parallel to the line AP (fig. 6) determined by $y = mx$; if therefore we take AB or AB' equal to c, according as c is affected with a positive or negative sign, and draw BD, or $B'D'$ parallel to AP, we shall have the straight line required. If m be negative, so that the pro-

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posed equation is $y = -mx + c$, then we must take AB or AB' (fig. 7) equal to c, and draw BD or $B'D'$ parallel to AP.

18. As the straight line is determined when any two points are known through which it passes, the position of the line which is the locus of any indeterminate equation of the first degree, may also be assigned by determining two of its points; and for this purpose the points most convenient are those in which it cuts the axes of x and y; the co-ordinates of which are obtained by making x and y successively zero in the given equation.

Thus if the equation be $y = mx + c$, in which m and c are positive, making $x = 0$, we have $y = AB = c$ (fig. 5), and making $y = 0$ we have $x = AT = -\frac{c}{m}$. Hence joining TB and producing it indefinitely, we have the line required.

The distance of the points of intersection from the origin, determined in this manner, must of course be measured along the positive or negative parts of the co-ordinate axes, accordingly as they are affected with positive or negative signs.

Problems relative to the straight line.

19. These principles being laid down, we proceed to the resolution of several problems relative to the straight line, the results of which are of great use; as its equation contains two disposable constants, they may be determined so as to make the line fulfil various conditions; as, for instance, to pass through two given points, to pass through a given point and be parallel, or perpendicular, or inclined at a known angle, to a given line, and so on.

20. To find the equation to a straight line which shall pass through two given points, whose co-ordinates are x', y', x'', y''.

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Any point in the line, of which the co-ordinates are x and y, being assumed, we have $y = mx + c$. But since x' and y' are also co-ordinates of a point in the same line, they will satisfy this equation,

$$\therefore\ y' = mx' + c.$$

Hence subtracting this equation from the former, we have

$$y - y' = m(x - x').$$

This is the equation to a line fulfilling one condition, viz. passing through the point (x', y'), and since m is arbitrary, an infinite number of lines may be so drawn. But since the line is moreover to pass through the point (x'', y''),

$$\therefore\ y'' - y' = m(x'' - x'),\ \text{ or } m = \frac{y'' - y'}{x'' - x'};$$

hence substituting this value of m in the above equation, we get

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

the equation required, which also may be written

$$y = \frac{y'' - y'}{x'' - x'}x + \frac{y'x'' - x'y''}{x'' - x'},$$

which is of the general form $y = mx + c$.

21. To find the equation to a straight line which shall pass through a given point and be parallel to a given straight line.

Let the equation to the given line be

$$y = mx + c,$$

where m and c are known; and the equation to the required line

$$y = m'x + c',$$

where m' and c' are unknown; then in order that these lines may be parallel we must have $m = m'$, for they are

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respectively parallel to lines passing through the origin whose equations are

$$y = mx, \quad y = m'x,$$

and these two latter lines must coincide, since the two former are parallel; consequently $m = m'$. The equation to the line parallel to the given line then becomes

$$y = mx + c',$$

where c' remains indeterminate, since there is an infinite number of lines which are parallel to a given line; but if it be required to pass through a given point (x', y') we must have

$$y' = mx' + c', \quad \text{which gives } c'.$$

Subtracting therefore this from the preceding, c' will disappear, and we have for the equation to the required parallel

$$y - y' = m(x - x').$$

22. Having given the equations to two straight lines, to determine their point of intersection.

Let the equations to the two lines be

$$y = mx + c.$$

$$y = m'x + c'.$$

At the point where these lines intersect, they have the same co-ordinates, and conversely their co-ordinates are not equal at any other point except that in which they intersect; hence for that point only we have

$$mx + c = m'x + c',$$

which gives

$$x = \frac{c - c'}{m' - m},$$

and substituting this value for x in the equation to one of the lines, we get

$$y = \frac{m'c - mc'}{m' - m}.$$

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When $m' = m$ these values become infinite, as ought to happen, for the lines are then parallel; when $c' = c$ as well as $m' = m$, the values become $\frac{0}{0}$, that is to say, indeterminate, which likewise ought to happen, as the lines then become coincident in all their points.

23. Having given the equations to two lines, to find the angle between them.

$$\text{Let } y = mx + c,$$

$$y = m'x + c',$$

be the equations to the lines BT, $B'T'$, (fig. 8); and α, α', the angles at which they are respectively inclined to the axis of x, so that $\tan \alpha = m$, $\tan \alpha' = m'$; and let ϕ be the angle between them.

Then angle $BPB' = \angle PTA - \angle PT'A$,

$$\text{or } \phi = \alpha - \alpha'; \ \therefore \ \tan \phi = \frac{\tan \alpha - \tan \alpha'}{1 + \tan \alpha . \tan \alpha'},$$

$$\text{or } \tan \phi = \frac{m - m'}{1 + mm'}.$$

Similarly, $\sin \phi = \sin \alpha . \cos \alpha' - \cos \alpha . \sin \alpha' = \frac{m - m'}{\sqrt{1+m^2} . \sqrt{1+m'^2}}$,

$$\cos \phi = \frac{1 + mm'}{\sqrt{1 + m^2} . \sqrt{1 + m'^2}}.$$

24. Hence, in order that the two lines may be parallel, we must have $\tan \phi = 0$, or $m - m' = 0$, as before.

And in order that they may be at right angles to one another, we must have $\tan \phi = \infty$, or $1 + mm' = 0$.

25. To find the equation to a straight line which passes through a given point, and makes a given angle with a given straight line.

Let BT (fig. 8) be the given line, and $y = mx + c$ its equation, therefore $\tan PTA = m$.

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$B'T'$ the required line whose equation may be assumed to be $y - y' = m'(x - x')$, since it passes through the point (x', y'), then tan $PT'A = m'$, also let tan $TPT' = t$ a given quantity; then because

$$\angle PT'A = PTA - TPT', \text{ or } \alpha' = \alpha - \phi,$$

$$\tan \alpha' = \frac{\tan \alpha - \tan \phi}{1 + \tan \alpha . \tan \phi}, \text{ or } m' = \frac{m - t}{1 + mt};$$

therefore $y - y' = \dfrac{m - t}{1 + mt}(x - x')$ is the required equation.

26. If the lines are to be parallel, $t = 0$, and as before the equation is

$$y - y' = m(x - x').$$

If the lines are to be perpendicular to one another $t = \infty$, and therefore the equation is

$$y - y' = -\frac{1}{m}(x - x').$$

Hence if the coefficient of x in one equation be the reciprocal of the coefficient of x in the other with a contrary sign, that is, if the equations be $y = mx + c$, $y = -\dfrac{1}{m}x + c'$, the lines which they represent are at right angles to one another; and in that case the co-ordinates of their points of intersection are (Art. 22),

$$x = \frac{m(c' - c)}{1 + m^2}, \quad y = \frac{c + m^2 c'}{1 + m^2}.$$

27. Having given the co-ordinates of a point and the equation to a straight line; to find the length of the perpendicular dropped from the point upon the line.

Let x', y', be the co-ordinates of the given point, and $y = mx + c$ the equation to the given line, then the equation to the perpendicular will be

$$y - y' = -\frac{1}{m}(x - x').$$

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In order to get the co-ordinates of the point of intersection of the given line and perpendicular, we must deduce from their equations values of x and y; to make the elimination more easy, put the first $y = mx + c$ under the form

$$y - y' = m(x - x') + c + mx' - y';$$

combining this with the equation to the perpendicular, and taking for the unknown quantities, the differences $y - y'$, $x - x'$, we get

$$x - x' = \frac{m(y' - mx' - c)}{1 + m^2}, \qquad y - y' = -\frac{y' - mx' - c}{1 + m^2},$$

values from which it is easy to deduce the co-ordinates x and y of the foot of the perpendicular. But if we denote by p the length of the perpendicular intercepted between the point and the given line, we have

$$p = \sqrt{(x - x')^2 + (y - y')^2};$$

therefore, putting for $x - x'$ and $y - y'$ their values,

$$p = \pm \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

As the value of p must be positive, we must take the upper or lower sign, according as the numerator $y' - mx' - c$ is positive or negative.

If the given point is situated in the origin, $x' = 0$, $y' = 0$, and the value of p is reduced to

$$p = \pm \frac{c}{\sqrt{1 + m^2}}.$$

28. The result of the preceding Article may be readily obtained as follows.

Let $y = mx + c$ be the equation to the given line MT (fig. 9),

$AN = x'$, $PN = y'$ the co-ordinates of the point P,

then the perpendicular $PQ = PR \cos RPQ = PR \cos RTN$.

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But $PN = y'$, and $RN = mx' + c$;

$$\therefore\ PR = y' - mx' - c;$$

$$\text{also } \cos RTN = \frac{1}{\sqrt{1 + \tan^2 RTN}} = \frac{1}{\sqrt{1 + m^2}};$$

$$\therefore \text{ the perpendicular } p = \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

29. Hence also, if through a given point a line be drawn cutting a given line at a known angle, we can find the distance of the given point from the point of intersection of the lines. For if PS be a line passing through the point P and cutting the line BM (fig. 9) at an angle $PSR = \alpha$, drawing PQ perpendicular to BM, we have

$$SP \sin \alpha = QP = \frac{y' - mx' - c}{\sqrt{1 + m^2}};$$

$$\therefore\ SP = \frac{y' - mx' - c}{\sin \alpha \sqrt{1 + m^2}}.$$

Straight Line referred to Oblique Co-ordinates.

30. To find the equation to a straight line referred to oblique co-ordinates.

Let the axes of the co-ordinates be inclined to one another at an angle ω, and suppose the line PT to cut the axis of x at an angle $= \alpha$. Let $AN = x$, $NP = y$, be co-ordinates of any point P, and draw BQ parallel to AX (fig. 10), meeting the ordinate of P in Q;

$$\text{then } \frac{PQ}{BQ} = \frac{\sin PBQ}{\sin BPQ} = \frac{\sin \alpha}{\sin (\omega - \alpha)}, \quad \text{or } PQ = \frac{x \sin \alpha}{\sin (\omega - \alpha)},$$

$$\text{and } NQ = AB = c;$$

$$\therefore\ y = \frac{x \sin \alpha}{\sin (\omega - \alpha)} + c,$$

the required equation.

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Hence if a straight line referred to oblique axes, be represented by the equation

$$y = mx + c,$$

m, the coefficient of x, expresses the ratio of the sines of the angles which the line makes respectively with the axes of x and y; and c, as before, is the ordinate through the origin.

In using the equation

$$y = \frac{x \sin \alpha}{\sin (\omega - \alpha)} + c,$$

we must remember, with respect to the constants involved, (1) that c is a positive or negative quantity, according as the line cuts the axis of y above or below the origin; (2) that ω is the angle YAX formed by the positive parts of the co-ordinate axes, and not the adjacent angle YAX'; and (3) that α is the angle PTX formed by the portion of the line which is situated above the axis of x, with the positive part of that axis.

31. It is easily seen that when a line is determined by the portions of the co-ordinate axes intercepted between it and the origin, its equation is of precisely the same form as when the co-ordinates are rectangular; for let $AB = c$, $AT = -a$, and x and y be the co-ordinates of any point P; we get from the similar triangles ABT, PBQ

$$\frac{c}{-a} = \frac{y - c}{x}, \quad \text{or} \quad \frac{x}{a} + \frac{y}{c} = 1.$$

See page 8.

Also when a line is determined by the condition of passing through two given points, or of being parallel to a given line, its equation is of the same form whether the co-ordinates be rectangular or oblique; in the following cases the results are different.

32. To find the angle between two lines whose equations are given, referred to oblique axes.

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First, to calculate the angle which a line whose equation is given, makes with the axis of x. Let $y = mx + c$ be its equation, and ω the angle of inclination of the axes, and let the line make an angle α with the axis of x, and therefore an angle $\omega - \alpha$ with the axis of y, then

$$\frac{\sin \alpha}{\sin (\omega - \alpha)} = m,$$

$$\text{which gives } \tan \alpha = \frac{m \sin \omega}{1 + m \cos \omega}.$$

Next, let $y = m'x + c'$ be the equation to another line referred to the same axes, making an angle α' with the axis of x;

$$\therefore \tan \alpha' = \frac{m' \sin \omega}{1 + m' \cos \omega}.$$

Let ϕ be the angle between the lines,

$$\text{then } \tan \phi = \tan (\alpha' - \alpha) = \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha' . \tan \alpha}$$

$$= \frac{(m' - m) \sin \omega}{1 + mm' + (m + m') \cos \omega}.$$

33. Hence if $\phi = 90^0$, then

$$1 + mm' + (m + m') \cos \omega = 0;$$

$$\therefore m' = - \frac{1 + m \cos \omega}{m + \cos \omega},$$

the condition in order that two lines referred to oblique axes, may be perpendicular to one another.

34. To find the perpendicular distance of a given point from a given line referred to oblique axes.

Let x', y', be the co-ordinates of the given point Q (fig. 11), $y = mx + c$ the equation to the given line CN which makes an angle α with the axis of x; and let QP

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be the perpendicular let fall from Q upon CN; then $QP = QN \cdot \sin(\omega - \alpha)$

$$= \frac{y' - mx' - c}{m} \sin\alpha = \frac{(y' - mx' - c)\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}},$$

since $\tan\alpha = \dfrac{m\sin\omega}{1 + m\cos\omega}$.

Straight Line referred to Polar Co-ordinates.

35. To find the polar equation to a straight line.

Let A be the pole (fig. 12), XA the initial line PT the proposed straight line, and $AP = r$, $XAP = \theta$ the polar co-ordinates of any point P in it; AQ a perpendicular upon it from the pole equal p, and XAQ the angle which that perpendicular makes with the initial line equal α,

then $AP = AQ \sec QAP$,

or $r = p \sec(\theta - \alpha)$.

If the proposed line be perpendicular to the initial line so that $\alpha = 0$, the equation becomes

$$r = p \sec\theta.$$

36. Since the polar equation to a straight line

$$r = p \sec(\theta - \alpha)$$

becomes $r\cos(\theta - \alpha) = p$,

or $r\cos\theta \cdot \cos\alpha + r\sin\theta \cdot \sin\alpha = p$,

it appears that every equation of the form

$$Ar\cos\theta + Br\sin\theta + C = 0,$$

is the polar equation to a straight line.

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$MA' = k$, the co-ordinates of the new origin A', $A'N' = x'$, $N'P = y'$, the co-ordinates of the same point P referred to the new axes $A'X'$, $A'Y'$, parallel to the former; then,

$$x = AM + A'N' = h + x' \text{ and } y = A'M + N'P = k + y',$$

and if these values be substituted for x and y, we shall have the equation to the curve referred to the origin A', the axes retaining their former directions.

40. To change the directions of the axes without altering the origin, supposing both systems to be rectangular.

Let $XAX' = YAY' = \theta$ (fig. 16) be the angle at which the new axes AX', AY' are inclined to the original axes AX, AY.

Let $AN = x$, $PN = y$, $AN' = x'$, $PN' = y'$, be the co-ordinates of the same point P referred to these respective axes.

Draw $N'Q$, $N'M$ parallel and perpendicular to AX,

$$\text{then } x = AM - QN' = x' \cos\theta - y' \sin\theta,$$

$$y = MN' + PQ = x' \sin\theta + y' \cos\theta,$$

and if these values be substituted for x and y, we shall have the equation to the curve referred to the axes AX', AY'.

41. To change the direction of the axes without altering the origin, supposing both systems to be oblique.

Let AX, AY (fig. 17) be the primitive axes inclined to one another at an angle $XAY = \omega$, and AX', AY' the new axes determined by the angles $X'AX = \alpha$, $Y'AX = \beta$, which they respectively form with the primitive axis of x.

Let P be any point, $AN = x$, $NP = y$ its primitive co-ordinates, and $AN' = x'$, $N'P = y'$ its new co-ordinates. Then denoting by xy the angle XAY contained by the

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axes of x and y produced in the positive directions, and similarly of the others, we have

$$x \sin xy = x' \sin x'y + y' \sin y'y,$$

each of these expressing the perpendicular distance of P from AY;

$$\therefore\ x = x' \frac{\sin(\omega - \alpha)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega}.$$

Also, $y \sin xy = x' \sin x' x + y \sin y' x$,

each of these expressing the perpendicular distance of P from AX;

$$\therefore\ y = x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega}.$$

In making use of these formulæ, we must recollect that ω represents the angle XAY contained by those portions of the primitive axes along which the positive co-ordinates are measured, and varies from zero to π; and that α and β denote the angles formed by AX', AY', the positive directions of the new axes, with the positive part AX of the primitive axis of x; and that each of these may receive any value from zero to 2π. See p. 111 art 230. the degree of an equation not altered by a change of axes

42. These formulæ are not often employed in the above general state; and we may deduce from them, as particular cases, the following, the use of which is more frequent.

First, by making $\omega = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2} + \alpha$, we fall upon the formulæ of the second case for passing from one system of rectangular co-ordinates to another system also rectangular.

Secondly, by making $\omega = \frac{\pi}{2}$ we obtain the formulæ for passing from a rectangular to an oblique system of co-ordinates, which are

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$$\left| \begin{aligned} x &= x' \cos \alpha + y' \cos \beta, \\ y &= x' \sin \alpha + y' \sin \beta. \end{aligned} \right.$$

Thirdly, by making $\beta = \frac{\pi}{2} + \alpha$ we obtain formulæ for passing from an oblique to a rectangular system of co-ordinates, which are

$$x = \frac{x' \sin (\omega - \alpha)}{\sin \omega} - \frac{y' \cos (\omega - \alpha)}{\sin \omega},$$

$$y = \frac{x' \sin \alpha}{\sin \omega} + \frac{y' \cos \alpha}{\sin \omega}.$$

43. In the preceding transformations we have supposed the origin to remain unaltered; if, however, the origin is to be changed as well as the direction of the axes, we must employ the formulæ

$$x = x'' + h, \qquad y = y'' + k,$$

where h, k are the co-ordinates of the new origin parallel to the primitive axes, and x'', y'' denote the values of x and y, found in each of the preceding cases.

44. To transform rectangular into polar co-ordinates, and conversely. *See fig. 77 vol 2.*

Let $AN = x$, $PN = y$ (fig. 4) be the rectangular co-ordinates of any point P in a curve referred to the rectangular axes AX, AY; and let $AP = r$, $PAX = \theta$ be the polar co-ordinates of P.

Then $AN = AP \cos PAN$, or $x = r \cos \theta$,

$$PN = AP \sin PAN, \qquad y = r \sin \theta,$$

and any equation between x and y may be transformed into the polar equation by substituting these values for x and y.

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Also since $r = \sqrt{x^2 + y^2}$, and $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$, any equation between r and θ may be transformed to rectangular co-ordinates, by substituting these values of r and θ.

If the pole do not coincide with the origin of rectangular co-ordinates, then if h and k be its co-ordinates, the quantities to be substituted for x and y, in order to get the polar equation will be

$$x = h + r\cos\theta, \quad y = k + r\sin\theta.$$

See p. 68 et. seq. Vol 2.

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SECTION IV.

ON THE CIRCLE.

Equation to a circle under various forms.

45. To find the equation to a circle when referred to two diameters at right angles to one another as axes.

Let P be any point in the circumference (fig. 18), $CN = x$, $NP = y$, its co-ordinates, and CP the radius equal c, then from the right-angled triangle CNP,

$$CN^2 + NP^2 = CP^2,$$

$$\text{or } x^2 + y^2 = c^2,$$

which is true for every point in the circumference, and is, therefore, the required equation; it expresses that the distance of every point from the origin is equal to c.

46. To find the equation to a circle when referred to any rectangular axes.

Let C be the center, and P any point in the circumference (fig. 19), draw CB, PN perpendicular to AX, and CM parallel to AX, and let the co-ordinates of C be $AB = a$, $BC = b$; the co-ordinates of P, $AN = x$, $NP = y$; and the radius $CP = c$.

Then from the right-angled triangle CPM,

$$CM^2 + MP^2 = CP^2.$$

But $CM = BN = x - a$, $MP = PN - CB = y - b$;

therefore we get for the general equation to the circle referred to rectangular axes,

$$(x - a)^2 + (y - b)^2 = c^2.$$

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47. From this we may deduce several particular forms of the equation to a circle which are worthy of notice.

First, putting $a = 0$, $b = 0$, we have the origin in the center and fall upon the equation already found,

$$x^2 + y^2 = c^2.$$

Secondly, putting $a = c$, $b = 0$, we have the origin at the extremity of a diameter, and that diameter the axis of x, and we get

$$(x - c)^2 + y^2 = c^2,$$

or, reducing,

$$y^2 = 2cx - x^2.$$

Thirdly, putting $b = 0$ simply, the center will be in the axis of x, but the origin will not be in the circumference; similarly, putting $a = 0$, the center will be in the axis of y, and in these two positions the equations will be

$$(x - a)^2 + y^2 = c^2,$$

$$x^2 + (y - b)^2 = c^2.$$

48. The above general equation to the circle when developed assumes the form

$$x^2 + y^2 + Ax + By + C = 0,$$

which does not contain the rectangle xy, and where the coefficients of the squares are equal. When these conditions are satisfied (the axes always being rectangular) the equation cannot represent any other curve except a circle. In fact by completing the squares we get

$$(x + \tfrac{1}{2}A)^2 + (y + \tfrac{1}{2}B)^2 = \tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C,$$

which evidently represents a circle the ordinates of whose center are $-\tfrac{1}{2}A$, $-\tfrac{1}{2}B$; and of which the radius

$$= \sqrt{\tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C}.$$

The equation however, will not in reality represent a circle, unless the quantity $\tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C$ is positive; if this quantity is zero the circle is reduced to a point, namely the center; if it is negative the equation is impossible. Peculiarities of this sort are offered by the equations

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$$x^2 + y^2 - 8y - 12x + 52 = 0,$$

$$x^2 + y^2 - 4y + 2x + 9 = 0,$$

which may be respectively reduced to the forms

$$(x-6)^2 + (y-4)^2 = 0, \quad (x+1)^2 + (y-2)^2 = -4.$$

But the equation $x^2 + y^2 + 4y - 4x - 8 = 0$, by completing the squares, becomes $(x-2)^2 + (y+2)^2 = 16$, which represents a circle whose radius is 4, and the co-ordinates of the center 2 and -2.

49. To find the equation to the circle referred to oblique axes.

Let, as before, $AB = a$, $BC = b$, $AN = x$, $NP = y$ (fig. 20) be the co-ordinates of the center, and of any point in the circumference of a circle referred to the oblique axes AX, AY, which form with one another an angle $= \omega$. Draw CM parallel to AX; then from the triangle CPM in which $\angle CMP = \pi - \omega$, and whose sides are respectively equal to $x - a$, $y - b$, and c the radius, we get

$$(x-a)^2 + (y-b)^2 + 2(x-a).(y-b)\cos\omega = c^2,$$

the required equation.

50. The above equation when developed becomes

$$x^2 + y^2 + 2xy\cos\omega - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y$$
$$+ a^2 + b^2 + 2ab\cos\omega - c^2 = 0,$$

which is of the form

$$x^2 + y^2 + 2xy\cos\omega + Ax + By + C = 0.$$

Whenever, therefore, an equation of the second order between oblique co-ordinates of known inclination, is such that the two squares x^2 and y^2 have unity for coefficients, and the rectangle xy has for coefficient twice the cosine of the angle between the axes, the equation represents a circle, the co-ordinates of whose center, a and b, and the radius c, are

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determined by the equations $a + b\cos\omega = -\frac{1}{2}A$, $b + a\cos\omega = -\frac{1}{2}B$, $a^2 + b^2 + 2ab\cos\omega - c^2 = C$.

Also dropping the perpendiculars CD, CE (fig. 20) upon the axes, we get

$$AD = a + b\cos\omega = -\tfrac{1}{2}A, \quad AE = b + a\cos\omega = -\tfrac{1}{2}B.$$

If therefore we take the distances AD, AE equal respectively to half the coefficients of x and y with contrary signs, and erect the perpendiculars CD, CE, we shall determine the position of the center C.

Thus in order that the equation

$$x^2 + xy + y^2 - 2ax - 2ay + a^2 = 0$$

may represent a circle, we must have $2\cos\omega = 1$ or $\omega = \frac{1}{3}\pi$; and if x', y' be the co-ordinates of its center, and c its radius,

$$x' + y'\cos\omega = a, \quad y' + x'\cos\omega = a, \quad x'^2 + y'^2 + 2x'y'\cos\omega - c^2 = a^2;$$

$$\therefore \quad x' = y' = \frac{2a}{3} \text{ and } c = \frac{a}{\sqrt{3}}.$$

51. To find the polar equation to a circle.

Taking the pole for the origin, let a, b be co-ordinates of the center of the circle (fig. 21), c its radius, and $AP = r$, $XAP = \theta$, the polar co-ordinates of any point P, supposing the initial line to coincide with the axis of x.

Then substituting $r\cos\theta$ for x and $r\sin\theta$ for y in the equation

$$(x - a)^2 + (y - b)^2 = c^2 \text{ and expanding we have}$$

$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta - 2br\sin\theta + b^2 = c^2,$$

$$\text{or} \quad r^2 - 2(a\cos\theta + b\sin\theta)r + a^2 + b^2 - c^2 = 0,$$

the required equation.

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52. If we suppose the pole to be in the circumference, and the initial line to be a diameter, we have $b = 0$, $a = c$, and the equation becomes

$$r^2 - 2cr\cos\theta = 0;$$

$$\therefore \quad r = 2c\cos\theta,$$

at which we arrive immediately by joining AP, PB (fig. 18), for the right-angled triangle BAP gives

$$AP = AB\cos BAP, \quad \text{or} \quad r = 2c\cos\theta.$$

53. Let PY be a tangent to the circle at P (fig. 21), and AY a perpendicular upon it from the pole; and suppose $AY = p$, $AC = d$, then

$$\frac{p}{r} = \sin APY = \cos APC = \frac{r^2 + c^2 - d^2}{2rc},$$

or $2cp = r^2 + c^2 - d^2$ a relation between r and p.

If the pole be in the circumference, or $c = d$, this becomes

$$r^2 = 2cp.$$

54. From the preceding equations to the circle, which assume no other property of a circle than that it is the locus of a point which is always at the same distance from a given fixed point, all the theorems relative to the circle established in geometry may readily be deduced. We shall however confine our attention to those which relate to the tangent.

Tangent and Normal to a Circle.

55. To find the equation to a straight line which touches a circle at a proposed point.

In geometry a line is said to touch a circle when it has only one point in common with the circumference; if therefore through the two points P, P' (fig. 22), we draw a

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secant PP', and then make it turn about P, till P' coincides with P, the secant in its ultimate position will become a tangent at P, for it will have only one point in common with the circumference. This consideration furnishes an easy method of determining the tangent at a given point of the circumference.

Let the co-ordinates of P, P', be respectively x', y'; x'', y'', and let α' be the angle which the line joining them makes with the axis of x,

$$\text{then } \tan\alpha' = \frac{y' - y''}{x' - x''};$$

but since the points are in the circumference, their co-ordinates must satisfy the equation to the circle;

$$\therefore \quad y'^2 + x'^2 = c^2, \quad y''^2 + x''^2 = c^2,$$

and subtracting the latter from the former we get

$$y'^2 - y''^2 + x'^2 - x''^2 = 0,$$

$$\text{which gives } \frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''},$$

$$\text{and consequently } \tan\alpha' = -\frac{x' + x''}{y' + y''}.$$

Now let $x'' = x'$ and $y'' = y'$ so that P' coincides with P, and the secant PT' assumes the position of the tangent PT; therefore denoting by α the angle which the tangent forms with the axis of x, we get

$$\tan\alpha = -\frac{x'}{y'}, \text{ and } y - y' = -\frac{x'}{y'}(x - x')$$

for the equation to the tangent; or multiplying by y', and remembering that $x'^2 + y'^2 = c^2$, the equation to the line touching the circle at the point x', y' is in its most simple form

$$yy' + xx' = c^2.$$

56. The equation to CP is $y = \frac{y'}{x'}x$, which compared with the above equation shews that CP and PT' are at right

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angles to one another; that is, the tangent to a circle at any point, and the radius drawn to the point of contact, are perpendicular to one another.

57. To find the equation to the normal at any point of a circle.

A line drawn through the point of contact perpendicular to the tangent is called a normal. The co-ordinates of the point being x', y', the equation to the normal will be of the form

$$y - y' = m'(x - x'),$$

the condition of being perpendicular to the tangent gives

$$m' = -\frac{1}{m} = \frac{y'}{x'} \quad \text{(Art. 24)};$$

$$\therefore \quad y - y' = \frac{y'}{x'}(x - x'),$$

$$\text{or reducing,} \quad y = \frac{y'}{x'}x,$$

which is the equation to a line passing through the origin, in this case the center. Hence all normals to a circle pass through the center.

58. To find the equation to a line which touches the circle, and passes through a given point without the circle.

Let h, k, be the co-ordinates of the given point; and x', y', those of the point of contact which are unknown; when they are found we shall have for the equation to the tangent

$$yy' + xx' = c^2,$$

and as the tangent passes through the given point, its equation must be satisfied by the co-ordinates of that point,

$$\therefore \quad ky' + hx' = c^2,$$

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and since the point of contact is in the circumference,

$$y'^2 + x'^2 = c^2,$$

which are the two equations that serve to determine x' and y'.

59. When a problem, as in the present case, leads to two equations between the co-ordinates x' and y' of an unknown point, each of the equations taken separately gives a geometrical locus on which the point is placed; consequently if we construct the two loci, we shall have two lines, the intersections of which will determine the points which satisfy the problem.

The locus of the second equation is the proposed circle, the locus of the first is a straight line AB (fig. 23), which is constructed by taking $CA = \frac{c^2}{h}$, $CB = \frac{c^2}{k}$, and joining AB; then the points T, T' in which this line cuts the circle are the points required.

Either of the equations may be replaced by another which results from combining them in any manner; and in solving problems in this way we must always select the combinations whose loci are easiest to construct.

Thus if we subtract the above equations we get

$$y'^2 - y'k + x'^2 - x'h = 0, \text{ or } (y' - \tfrac{1}{2}k)^2 + (x' - \tfrac{1}{2}h)^2 = \tfrac{1}{4}k^2 + \tfrac{1}{4}h^2,$$

which represents a circle whose center is O the middle point of CP, and radius CO, P being the point through which the tangents are to be drawn; if then we join CP and bisect it in O (fig. 23), and with center O and radius OC describe a circle cutting the former in T, T', these are the points of contact, and are determined by a simpler construction than the former one.

The value $AC = \frac{c^2}{h}$, which determines the point A, in which TT' meets the axis of x, is independent of the

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ordinate k of P; therefore A will remain in the same position for all positions of P in the indefinite line PM. If therefore from every point of a straight line we draw tangents to a circle and join the points of contact, all the secants will intersect in the same point.

60. If we consider two circles represented by the equations

$$x^2 + y^2 = c^2, \quad (x-a)^2 + y^2 = c'^2,$$

so that the axis of x joins the centers, we may easily prove the co-ordinates of their points of intersection to be

$$x = \frac{a^2 + c^2 - c'^2}{2a},$$

$$y = \pm \frac{1}{2a}\sqrt{(a+c+c').(a+c-c').(a+c'-c).(c+c'-a)},$$

from which all the common theorems in geometry, relative to the intersections and contacts of circles with one another may be deduced.

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SECTION V.

ON THE DIFFERENT ORDERS OF CURVES; AND ON THE DIVISION OF CONIC SECTIONS, OR CURVES OF THE SECOND ORDER, INTO THREE SPECIES.

61. LINES are divided into orders according to the degree of their equations, the degree being determined by the sum of the indices of x and y in that term of the equation (which is supposed to contain no fractional or irrational term) where it is greatest.

The straight line is the line of the first order, being the locus of the equation of the first degree between two variables; the circle is a line of the second order, or curve of the second order (these terms being used indifferently), because its equation is of the second degree.

62. Curves of the second order are those whose equations involve the squares, or the simple product of the variables x and y, but no powers or products of them which are of higher dimensions. Hence the equation to curves of the second order under its most general form, or, which is the same thing, the general equation of the second order between two variables, is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

which (as will be hereafter shewn) by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

representing two distinct families of curves; the former those which have a center, the latter those which have not a center.

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Thus let S (fig. 25) be the given fixed point, and KX the line given in position, P a point such that joining SP and drawing PM perpendicular to KX, the distances SP, PM, are always to one another in an invariable ratio, then the locus of P is a Conic Section.

The point S is called the focus, and the line KX the directrix. Since SP may be always equal to PM, or always less than PM in a constant ratio, or always greater than PM in a constant ratio, there will be a distinct species of Conic Section corresponding to each of these cases; in the first case the locus of P is called the Parabola, in the second the Ellipse, and in the third the Hyperbola. sup. 55 p. 76

Note. The polar of either focus is the directrix of the Conic section

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SECTION VI.

ON THE PARABOLA.

Various forms of the equation to the Parabola.

69. To find the equation to the parabola.

The parabola is the locus of a point, whose distance from a given point is always equal to its distance from a given fixed line.

Let KX (fig. 26) be the given fixed line, and S the given point, from which draw SX perpendicular to KX, and bisect it in A; then A is a point in the curve; and since the distance SX is known, let $AS = a$.

Draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the rectangular axes of the co-ordinates; and let P be a point in the parabola, and $AN = x$, $PN = y$, its co-ordinates; then drawing PM perpendicular to KX, and joining SP,

$$SP^2 = PM^2,$$

$$\text{or } PN^2 + SN^2 = XN^2,$$

$$\text{or } y^2 + (x - a)^2 = (x + a)^2;$$

$$\therefore y^2 = 4ax,$$

the equation required.

70. To trace the parabola by means of its equation (fig. 26).

Solving the equation, we get $y = \pm 2\sqrt{ax}$,

which shews, since for each positive value of x there are two equal values of y with contrary signs, that the axis of x is

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an axis of the curve; and that the origin is a point in the curve, since $x = 0$, gives $y = 0$; and that no part of the curve is situated to the left of A, for a negative value of x makes y imaginary; but that as x increases from zero to infinity, y also increases from zero to $\pm\infty$. Moreover the curve is always concave toward its axis, otherwise it might be intersected in more than two points by a straight line, which is impossible. The parabola has only one vertex, namely, the point where it is met by the axis; and only one focus and directrix; and consists of two perfectly similar infinite branches Ax, Ax' upon the same side of the axis of y, and situated symmetrically with respect to the axis of x, to which they turn their concavities.

71. The double ordinate through the focus is called the latus rectum of the parabola; to find its length, making $x = AS = a$, we get

$$y^2 = 4a^2; \quad \therefore y = \pm 2a = SB \text{ or } SC;$$

and consequently $BC = 4a$.

Hence, if P be any point in the curve, we have

$$PN^2 = BC \times AN,$$

or, the square of the ordinate is equal to the rectangle of the latus rectum and corresponding abscissa, and consequently varies as the abscissa.

72. Let the origin be a point C (fig. 27) in the curve, and let the axis of the abscissæ be a line perpendicular to the axis of the parabola passing through C; and let $CM = x$, $MP = y$, be the co-ordinates of any point P, and $CB = h$, $AB = k$, those of the vertex; then

$$PN^2 = 4a \,.\, AN,$$

$$\text{or } (h - x)^2 = 4a\,(k - y),$$

$$\text{or } y = \frac{h}{2a}x - \frac{x^2}{4a}, \text{ since } h^2 = 4ak,$$

another form of the equation which is sometimes useful.

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73. To express the distance of any point in the parabola from the focus, in terms of its abscissa.

$$SP^2 = SN^2 + NP^2$$
$$= (x - a)^2 + 4ax = (x + a)^2;$$
$$\therefore SP = x + a.$$

74. In expressing, as above, the distance of any point in the parabola from an assumed point, it is only when the latter coincides with the focus that the expression becomes rational.

For let x', y', be the co-ordinates of the assumed point; x, y, those of the point in the parabola, and d their distance; then

$$d^2 = (x - x')^2 + (y - y')^2 = x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2,$$

but $y = 2\sqrt{ax}$, therefore d^2, and *a fortiori* d, cannot be rational in terms of x, unless the term $2yy'$ vanish, which gives $y' = 0$; then, replacing y^2 by its value,

$$d^2 = x^2 + 4ax - 2xx' + x'^2$$

which must be a perfect square;

$$\therefore 4x'^2 = 4(2a - x')^2,$$
$$x' = 2a - x',$$
$$x' = a,$$

which with the value $y' = 0$, indicates the focus; this then is the only point whose distance from any point in the curve can be expressed rationally in terms of the abscissa of that point. This is sometimes given as the definition of the focus.

75. To find the polar equation to the parabola, the the focus being the pole.

Let $SP = r$, angle $PSx = \theta$, (fig. 26);

$\therefore SN = r\cos\theta$ and $x = a + r\cos\theta$,

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$$\text{but } SP = a + x;$$

$$\therefore\ r = 2a + r\cos\theta,$$

$$\text{or } r = \frac{2a}{1 - \cos\theta}.$$

Sometimes the angle of revolution is measured from that part of the axis which passes through the vertex, in which case, if $ASP = \theta'$, putting $\pi - \theta'$ for θ, we get

$$r = \frac{2a}{1 + \cos\theta'}.$$

Tangent and Normal to the Parabola.

76. To find the equation to the tangent of a parabola at a given point.

Let x', y', be the co-ordinates of the given point P (fig. 28), and x'', y'', those of another point P' in the parabola near the former; if we draw a secant through these two points, and denote by α' the angle which it forms with the axis of x, we have

$$\tan\alpha' = \frac{y'' - y'}{x'' - x'}.$$

But the points being in the parabola, we have

$$y''^2 = 4ax'', \quad y'^2 = 4ax';$$

$$\therefore\ y''^2 - y'^2 = 4a(x'' - x'),$$

$$\frac{y'' - y'}{x'' - x'} = \frac{4a}{y'' + y'},$$

$$\tan\alpha' = \frac{4a}{y'' + y'}.$$

Now let P' move up to and coincide with P, then $x'' = x'$, $y'' = y'$, and the secant becomes the tangent at (x', y'); there-

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fore, denoting by α the angle PTN which the tangent makes with the axis of x, we get

$$\tan \alpha = \frac{2a}{y'}.$$

And the equation to the tangent is

$$y - y' = \frac{2a}{y'}(x - x')$$

or, multiplying by y' and observing that $y'^2 = 4ax'$,

$$yy' = 2a(x + x');$$

in which x', y', are the co-ordinates of the point of contact, and x, y, co-ordinates of any point in the tangent line.

77. If in the formula $\tan \alpha = \frac{2a}{y'}$ we make $y' = 0$, we find $\tan \alpha = \infty$, therefore the tangent at the vertex is perpendicular to the axis; also if we suppose y' to increase up to infinity, α decreases to zero; hence the tangent to the parabola continually tends to become parallel to the axis.

Hence, the equation of Art. 72. may be put under the form

$$y = x \tan \alpha - \frac{x^2}{4a}, \quad \text{if } \angle TCB = \alpha.$$

78. In the parabola, the subtangent, which is the distance between the foot of the ordinate to any point and the intersection of the tangent at that point with the axis, is double of the abscissa.

For $TN \times \tan PTN = PN$ (fig. 29),

$$\text{or } TN \times \frac{2a}{y'} = y'; \quad \therefore TN = \frac{y'^2}{2a} = 2x'.$$

This result may also be obtained by making $y = 0$ in the equation to the tangent; this gives $x = -x'$, and proves that the point T where the tangent meets the axis of x, is situated

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to the left of A, and at a distance $AT = AN$. Hence, adding AN to AT, we have the subtangent $TN = 2AN$.

This furnishes a simple construction for drawing a tangent to a parabola at a given point.

If P be the given point, and AN, NP its co-ordinates, we have only to take in NA produced, $AT = AN$, and join TP, then TP is the tangent required.

79. In any curve, a line drawn through the point of contact perpendicular to the tangent is called a normal; and the distance between the foot of the ordinate, and the intersection of the normal with the axis of x, is called the subnormal.

In the parabola, the subnormal is equal to half the latus rectum; for if PG be perpendicular to PT since in the triangle TPG (fig. 29), PN is drawn from the right angle perpendicular to the opposite side,

$$NG \times TN = PN^2, \text{ or } NG \times 2x = 4ax;$$

$$\therefore NG = 2a.$$

80. This result may also be obtained by finding the equation to the normal at the point (x', y') of the parabola.

It will be of the form

$$y - y' = m'(x - x');$$

and since it is perpendicular to the tangent whose equation is

$$y - y' = \frac{2a}{y'}(x - x'),$$

$$m' = -\frac{1}{m} = -\frac{y'}{2a} \quad \text{(Art. 26)};$$

therefore the equation to the normal is

$$y - y' = -\frac{y'}{2a}(x - x');$$

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now make $y = 0$, then $x - x' = 2a$,

or $AG - AN = NG = 2a$.

81. Since $ST = AS + AT = a + x$,

and $SG = SN + NG = x - a + 2a = a + x$,

we have $SP = ST = SG$ (Art. 73).

82. Draw Px' through P parallel to the axis, then

$$\angle tPx' = PTS = SPT, \text{ since } SP = ST;$$

also $\angle GPx' = SPG$.

Hence the tangent and normal at any point make equal angles with the focal distance of that point, and with a line drawn through it parallel to the axis.

83. These properties furnish a simple method of drawing a tangent to a parabola through a given point.

First, let the point be in the parabola, as P (fig. 29); join SP, and with center S and radius SP, describe a circle cutting the axis in T and G, then if PT and PG be joined, they are the tangent and normal at P.

84. Next, let the point be without the parabola, as T' (fig. 30), and with center T' and radius $T'S$ describe a circle cutting the directrix in two points M and M', through which draw two parallels to the axis, meeting the parabola in P, P', these are the points of contact; for the triangles MPT', $T'PS$ are equal in all respects, therefore $\angle T'PM = T'PS$, and PT' is a tangent at P; similarly, TP' is a tangent P'.

85. This problem may be also solved by means of the equation to the tangent, as in the case of the circle.

Let h, k, be the co-ordinates of the given external point, and x', y', those of the unknown point of contact; then since they must satisfy both the equation to the tangent and that to the curve, we have to determine them;

$$ky' = 2a(h + x'),$$

$$y'^2 = 4ax',$$

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and if we construct the straight line represented by the former, considering x' and y' as the variables, the points in which it intersects the parabola are the points of contact.

It appears that if a pair of tangents be drawn from an external point (h, k), the equation to the chord joining the points of contact is

$$ky = 2a(x + h).$$

86. The locus of the foot of the perpendicular dropped from the focus upon the tangent to a parabola is the line touching the parabola at its vertex.

Let PT (fig. 29) the tangent at P, meet Ay in Y, and join SY; then because TN is bisected in A, PT is bisected in Y; and since $SP = ST$, the triangles SPY, STY are equal in all respects; therefore SY is perpendicular to PT. Hence the tangent at any point and the perpendicular upon it from the focus intersect in the line which touches the parabola at the vertex.

87. Also from the right-angled triangle SYT we have

$$SY^2 = ST \times SA = SP \times SA,$$

or $p^2 = ar$, denoting SP, SY, by p and r respectively.

88. Let the tangent at P (fig. 31) meet the directrix in Q, draw PM perpendicular to the directrix, and join SQ; then $SP = PM$, PQ is common to the two triangles SPQ, QPM, and angle $SPQ = QPM$ by what has been proved;

$$\therefore \angle QSP = PMQ = \text{a right angle.}$$

Hence if a perpendicular to any focal distance SP intersect the directrix in Q, and QP be joined, QP is a tangent at P.

Therefore producing PS to meet the parabola in P' and joining QP', this is a tangent at P'; and since angle $PQS = PQM$ and angle $P'QS = P'QM'$, we have angle $PQP' = \frac{1}{2}\pi$. Hence the tangents at the extremities of any focal chord intersect at right angles in the directrix; and the line joining their point of intersection and the focus is perpendicular to the chord.

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The Parabola referred to any Diameter.

89. To determine the intersection of a straight line with a parabola.

Let $y = mx + c$, be the equation to any straight line; if this line intersect a parabola whose equation is $y^2 = 4ax$, and if x', y', be the co-ordinates of a common point, we have

$$y' = mx' + c, \quad y'^2 = 4ax';$$

$$\therefore\ y' = \frac{m}{4a}y'^2 + c, \quad \text{or } y'^2 - \frac{4a}{m}y' + \frac{4ac}{m} = 0,$$

the roots of which are the ordinates of the points where the straight line meets the curve. Hence a straight line cannot cut a parabola in more than two points; if the roots be equal, the points of section coincide, and the line is then a tangent, and we have $\frac{16ac}{m} = \frac{16a^2}{m^2}$, or $m = \frac{a}{c}$; if the roots be impossible, the line falls entirely without the parabola.

90. To find the locus of the middle points of a system of parallel chords.

Let QQ' (fig. 32) be any chord whose equation is $y = mx + c$, V its middle point; draw VM perpendicular to Ax, then

$$VM = Q'N' + \tfrac{1}{2}(QN - Q'N') = \tfrac{1}{2}(Q'N' + QN),$$

but the values of QN, $Q'N'$, are the roots of the equation

$$y^2 - \frac{4a}{m}y + \frac{4ac}{m} = 0,$$

obtained by eliminating x between the equations to the parabola and chord;

$$\therefore \quad QN + Q'N' = \frac{4a}{m};$$

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hence denoting the ordinate of V by Y, we have for the equation to the locus of V, $Y = \frac{2a}{m}$, which represents a straight line PV parallel to the axis; and since the equation does not involve c, it bisects all chords for which m is the same, that is, all chords parallel to QQ'.

91. The line PV is called a diameter of the parabola. All diameters of a parabola are parallel to the axis, and intersect the corresponding chords at different angles varying from zero to a right angle.

Conversely, every straight line parallel to the axis may be considered as a diameter to the parabola; for by giving m a suitable value in the equation $Y = \frac{2a}{m}$, Y may become equal to any quantity we please.

92. Suppose a diameter Px' to be drawn at a distance y' from the axis, we have for this diameter

$$\frac{2a}{m} = y', \text{ or } m = \frac{2a}{y'}.$$

The quantity m is the tangent of the angle at which the diameter in question meets the chords which it bisects; it is also (Art. 76) the value of the tangent of the angle which the line touching the parabola at P, makes with the axis of x; therefore the chords bisected by any diameter, are parallel to the tangent at the extremity of that diameter.

The semi-chord QV is called an ordinate to the diameter PV.

In order that m may be infinite, we must have $y' = 0$; hence the axis of x is the only diameter which bisects its ordinates at right angles, or is the only axis of the parabola.

93. To find the equation to the parabola when referred to any diameter, and the tangent at the extremity of the diameter, as axes.

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The formulæ for passing from rectangular to oblique axes are (Art. 42),

$$x = x' \cos \alpha + y' \cos \beta + h, \quad y = x' \sin \alpha + y' \sin \beta + k.$$

Suppose the new origin to be a point P (fig. 35) in the curve; then between h and k we have the relation $k^2 = 4ah$; also let the diameter Px' be the axis of x', then $\alpha = 0$, and if we take the tangent Py' for the axis of y', we have $\tan \beta = \frac{2a}{k}$.

Now substituting in the equation $y^2 = 4ax$, we get

$$(y' \sin \beta + k)^2 = 4a\,(x' + y' \cos \beta + h),$$

$$\text{or } y'^2 \sin^2 \beta + 2y'k \sin \beta + k^2 = 4ax' + 4ay' \cos \beta + 4ah,$$

$$\text{or } y'^2 \sin^2 \beta = 4ax', \text{ since } k^2 = 4ah, \text{ and } k \sin \beta = 2a \cos \beta.$$

$$\text{But } \frac{a}{\sin^2 \beta} = a\,(1 + \cot^2 \beta) = a\left(1 + \frac{k^2}{4a^2}\right) = a + h = SP,$$

$$\therefore \quad y'^2 = 4SP \,.\, x' = 4a'x', \text{ if } SP = a', \text{ or } QV^2 = 4SP \,.\, PV.$$

94. The coefficient $4SP$, by which one diameter differs from another, is called the parameter of the diameter to which the parabola is referred; it is equal to four times the distance of the focus from the extremity of the diameter. It is also equal to the double ordinate passing through the the focus. For draw QQ' (fig. 33) through the focus S parallel to the tangent PT; then $PV = ST = SP$,

$$\therefore \quad QV^2 = 4SP \times PV = 4SP^2;$$

$$\therefore \quad QV = 2SP, \text{ and } QQ' = 4SP.$$

95. The equation to a parabola being of the same form when referred to a diameter and the tangent at its extremity, as when referred to its axis, the properties which are independent of the inclination of the co-ordinates, will be the same in the two systems. Hence the equation to the tangent at a point Q (x', y') will be

$$yy' = 2a'(x + x'),$$

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where $\frac{2a'}{y'}$ denotes the ratio of the sines of the angles which the tangent makes with the axes of x and y; and when the tangent meets the axis of x we shall have $x = -x'$, or the subtangent $= 2x' =$ twice the abscissa of the point of contact, in all cases, i. e. $TV = 2\,PV$ (fig. 35).

96. Also if we wish to draw a tangent through an external point Q (h, k), we shall have, to determine the points of contact (x', y'), the equations

$$y'^2 = 4a'x', \quad y'k = 2a'(x' + h);$$

and if we construct the line represented by the latter, considering x' and y' as the variables, by taking $PT = -h$, $PR = \frac{2a'h}{k}$ (fig. 34), and joining TR, it will cut the parabola in the two points of contact.

97. Since the distance PT is independent of k, if through Q we draw a line parallel to Py', and from any other point in this line we draw a pair of tangents to the parabola, the secant passing through the new points of contact will cut the diameter Px' in T, as this point only changes when h changes. Hence if from the several points of a straight line, pairs of tangents be drawn to a parabola, the secants joining the corresponding points of contact will all intersect in the same point; and conversely, if through any point we draw different chords, and draw two tangents at the extremities of each, the locus of their intersection will be a straight line.

98. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking the diameter and the tangent at its extremity as axes, the equation to the tangent will be

$$\pm yy' = 2a'(x + x');$$

using the upper or lower sign according as we consider the point Q (x', y'), or the other extremity of the chord Q', whose

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co-ordinates are x', $-y'$; and in both cases $y = 0$ when $x = -x'$; therefore the tangents meet the diameter in the same point T (fig. 35).

99. Having given the parameter of any diameter of a parabola, and the inclination of the corresponding ordinates, to describe the parabola.

Let Px' be the given diameter (fig. 33), draw the line $y'PT$ at the given inclination to Px', this line will be a tangent to the parabola at the point P. Make the angle $TPS = y'Px'$, and PS a quarter of the given parameter; then S will be the focus. Take $PM = PS$ and draw ML perpendicular to Mx', this will be the directrix; and knowing the focus and directrix, the parabola can of course be described.

100. If a parabola be traced upon a plane, we may determine its axis by drawing two parallel chords QQ', qq', (fig. 35), and drawing a line VV' through their middle points, this will be a diameter. And if we draw any chord QR perpendicular to it, and through the middle point of QR draw AN parallel to VV', this will be the axis of the parabola.

101. If through any point within or without a parabola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.

Let O be the given point (fig. 36), Qq a line drawn through it in a known direction, and therefore an ordinate to a given diameter PV; draw the diameter $A'O$, and $A'v$ parallel to Qq; then

$$QO \times Oq = QV^2 - VO^2 = 4SP \times PV - 4SP \times Pv = 4SP \times A'O.$$

Similarly, if $Q'q'$ be an ordinate to the diameter whose extremity is P', passing through O,

$$Q'O \times Oq' = 4SP' \times A'O;$$

$$\therefore QO \times Oq : Q'O \times Oq' :: SP : SP',$$

a ratio independent of the position of O.

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102. If the point O be without the parabola, and we suppose Q, q, to coincide in P, and Q', q' to coincide in P', so that OP, OP' become tangents, we have

$$OP^2 : OP'^2 :: SP : SP'.$$

103. Any parabolic segment ANP is two-thirds of the parallelogram whose sides are the abscissa AN and the ordinate NP.

Let NP, $N'P'$ (fig. 37) be ordinates to the diameter AN; at P, P' draw tangents meeting the diameter in T, T', and one another in R. Join PP' and draw RK parallel to AN, meeting PP' in I, and bisecting it in that point (Art. 98); and draw KH perpendicular to AN.

Then area of triangle $TRT' = \frac{1}{2} TT' \times KH$,

area of trapezium $N'P'PN = NN' \times KH = TT' \times KH$,

since $AN = AT$, and $AN' = AT'$.

Hence the trapezium is double of the triangle. Similarly we may shew that trapezium $N''P''P'N'$ is double of the triangle $T'R'T''$, and so on. Hence the sum of the trapeziums is double of the sum of the triangles, and therefore the parabolic segment APN, which is the limit of the first sum, is double of the exterior segment APT, which is the limit of the second sum. Hence the segment ANP is two-thirds of the triangle TNP, or two-thirds of the parallelogram contained by AN, NP.

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SECTION VII.

ON THE ELLIPSE.

Various forms of the equation to the Ellipse.

104. To find the equation to the Ellipse.

The ellipse is the locus of a point, whose distance from a given point is always less than its distance from a given fixed line, in a constant ratio.

Let S (fig. 38) be the given point, and KK' the given fixed line; and from S let fall the perpendicular SX upon KK'. Let P be a point in the ellipse; join SP and draw PM perpendicular to KX; and let the constant ratio $SP : PM$ be $e : 1$, e being less than 1. Divide SX in A so that $SA : AX :: e : 1$, then A is a point in the ellipse; and since the distance SX is known, we may assume $AS = p$, therefore $AX = \frac{p}{e}$. Through A draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the axes of the co-ordinates; and let $AN = x$, $PN = y$, be the co-ordinates of P.

Then $SP^2 = e^2 . PM^2$,

or $SN^2 + NP^2 = e^2 . XN^2$,

or $(x - p)^2 + y^2 = e^2 \left(\frac{p}{e} + x\right)^2 = (p + ex)^2$;

$$\therefore y^2 = 2p(1 + e)x - (1 - e^2)x^2$$

$$= (1 - e^2) \cdot \left(\frac{2p}{1 - e}x - x^2\right),$$

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Since $b = a\sqrt{1-e^2}$, the eccentricity, expressed by the semi-axes, is equal to $\frac{\sqrt{a^2-b^2}}{a}$. Hence $SC = \sqrt{a^2-b^2}$, and $\therefore\ BS = a$; and if with center B and radius equal to the semi-axis major we describe a circle, it will intersect the major axis in the foci.

Hence each focus divides the major axis into the segments $a - \sqrt{a^2-b^2}$, $a + \sqrt{a^2-b^2}$, the rectangle of which equals the square of the semi-axis minor.

110. Since $AS = e \cdot AX$, we have $AX = \frac{a(1-e)}{e}$;

$$\therefore\ CX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS},$$

which determines the directrix relative to the center.

111. The double ordinate passing through the focus is called the latus rectum.

To find its value, make $x = CH = ae$ in the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\text{then } y^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = b^2(1-e^2) = \frac{b^4}{a^2};$$

$$\therefore\ y = \pm\frac{b^2}{a};$$

$$\therefore\ LL' = \frac{2b^2}{a},\ \text{or}\ = 2a(1-e^2).$$

112. When the distance CS between the focus and center becomes infinite, the distance AS between the focus and vertex remaining finite, the ellipse is changed into a parabola. For since $\frac{b^2}{a^2} = 1 - e^2$, the equation reckoned from the vertex may be written

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$$y^2 = 2a(1-e^2)x - (1-e^2)x^2,$$

$$\text{or } y^2 = 2p(1+e)x - (1-e^2)x^2, \quad \text{if } AS = p.$$

$$\text{But } e = \frac{SC}{AC} = \frac{SC}{p+SC} = \frac{1}{1+\frac{p}{SC}}; \quad \text{let } SC = \infty; \quad \therefore e = 1;$$

$$\therefore y^2 = 4px, \quad \text{the equation to the parabola.}$$

113. When $a = b$ the equations to the ellipse become

$$y^2 = a^2 - x^2, \quad y^2 = 2ax - x^2,$$

which represent a circle; hence when its axes are equal, the ellipse becomes a circle.

Upon the major axis as diameter describe a circle (fig. 39), and produce the ordinate NP of the ellipse to meet it in Q; then making $CN = x$, $NP = y$, $NQ = y'$, we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y' = \sqrt{a^2 - x^2};$$

$$\therefore y = \frac{b}{a}y';$$

which shews that, corresponding to the same abscissa, the ordinates of the ellipse are to the ordinates of the circle in the constant ratio of the smaller to the larger axis; consequently the ellipse may be described by diminishing all the ordinates of the circle in that ratio.

114. Since $y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}(a+x).(a-x)$,

$$\text{gives } PN^2 = \frac{b^2}{a^2} \times A'N \times AN \text{ (fig. 39)},$$

we see that the square of the ordinate is to the rectangle of the corresponding segments of the major axis, as the square of the semi-axis minor to the square of the semi-axis major; and that, consequently, the square of the ordinate varies as the rectangle of the corresponding segments of the major axis.

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115. To express the distances of any point in the ellipse from the foci, in terms of its abscissa

$$SP^2 = SN^2 + PN^2 \text{ (fig. 39)}$$

$$= (ae + x)^2 + (1 - e^2) \cdot (a^2 - x^2)$$

$$= (a + ex)^2;$$

$$\therefore SP = a + ex,$$

$$HP^2 = (ae - x)^2 + (1 - e^2) \cdot (a^2 - x^2)$$

$$= (a - ex)^2;$$

$$\therefore HP = \pm (a - ex),$$

and since e is less than 1, and x is always less than a, in order that HP may be positive we must take the upper sign which gives

$$HP = a - ex.$$

116. In expressing, as above, the distance of any point in the ellipse from an assumed point, it is only when the latter coincides with one of the foci, that the expression becomes rational.

For let x', y', be the co-ordinates of the assumed point, x, y, those of any point in the ellipse, and d their distance,

$$\text{then } d^2 = (x - x')^2 + (y - y')^2$$

$$= x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2.$$

But $y = \frac{b}{a}\sqrt{a^2 - x^2}$, therefore d^2, and *a fortiori* d, cannot be expressed rationally in terms of x unless the term $2yy'$ vanish, which gives $y' = 0$; then replacing y^2 by its value we get

$$d^2 = \left(1 - \frac{b^2}{a^2}\right) x^2 - 2xx' + x'^2 + b^2$$

which must be a perfect square;

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$$\therefore\ 4\left(1-\frac{b^2}{a^2}\right)(x'^2+b^2)=4x'^2,$$

$$\text{or } x'=\pm\sqrt{a^2-b^2}.$$

These values require that a should be greater than b, i. e. that the abscissæ should be measured along the axis major; and with $y'=0$, they determine, as we perceive, the two foci S and H. These then are the only points whose distances from any point of the curve can be expressed rationally in terms of the abscissa of the point. This is sometimes given as the definition of the foci.

117. Hence, by addition we get

$$SP+HP=2a,$$

which expresses the remarkable property, that the sum of the focal distances of any point in the ellipse is constant, and equal to the major axis.

This affords a very simple method of determining any number of points of an ellipse of which we know the foci and axis major; in AA' take any point F (fig. 40), and with center S and radius equal to $A'F$ describe a circle; next with center H and radius equal to AF describe another circle, cutting the former in P, P', these are manifestly points in the ellipse.

When the ellipse is to be very large, we may describe it by fastening the ends of a cord equal to the axis major in the foci; then if we make a pole slide along the cord, so as to keep it stretched, the ellipse will be traced out by the extremity of the pole.

This property also furnishes a method of investigating the equation to the ellipse which is sometimes employed as follows.

118. To find the locus of a point, the sum of whose distances from two given points is constant.

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Through the two fixed points S, H (fig. 40), draw the indefinite line Sx; bisect SH in C, and draw yC perpendicular to it, and take Cx, Cy, for the axes of the co-ordinates, as the locus of P will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point P, and $SP + PH = 2a$;

$$\text{then } SP^2 = (x + c)^2 + y^2,$$

$$HP^2 = (x - c)^2 + y^2;$$

$$\therefore SP^2 - HP^2, \text{ or } 2a(SP - HP) = 4cx;$$

$$\therefore SP - HP = \frac{2cx}{a},$$

$$\text{but } SP + HP = 2a;$$

$$\therefore SP = a + \frac{cx}{a};$$

$$\therefore \left(a + \frac{cx}{a}\right)^2 = (x + c)^2 + y^2,$$

$$\text{or } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2).$$

Now $SP + HP$ is greater than SH, or a is greater than c, therefore $a^2 - c^2$ is a positive quantity; hence comparing the above with the equation to an ellipse

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

we see that the equations are identical, and consequently so are the curves which they represent, if $b^2 = a^2 - c^2$; therefore the required locus is an ellipse whose major axis is $2a$, and minor axis $2\sqrt{a^2 - c^2}$.

119. To find the polar equation to the ellipse, one of the foci being the pole.

Let $SP = r$, $ASP = \theta$ (fig. 40); $\therefore SN = r\cos\theta$,

and $CN = r\cos\theta - ae$, but $SP = a + ex$;

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$$\therefore\ r = a + e(r\cos\theta - ae) = a(1-e^2) + er\cos\theta;$$

$$\therefore\ r(1 - e\cos\theta) = a(1-e^2),$$

$$\text{or } r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

We have measured the angle θ from that part of the axis major which passes through the more remote vertex; sometimes it is measured from the nearer vertex A', in which case if $A'SP = \theta'$, putting $\pi - \theta'$ for θ we get

$$r = \frac{a(1-e^2)}{1 + e\cos\theta'}.$$

Of course if the other focus H be taken for the pole, the formulæ will be exactly the same.

120. To find the polar equation to the ellipse, the center being the pole.

Let $CP = r$, $ACP = \theta$ (fig. 40), and let x and y be the rectangular co-ordinates of P; then $x = r\cos\theta$, $y = r\sin\theta$; therefore substituting these values in the equation

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ we get}$$

$$r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2,$$

or, dividing by a^2 and observing that $\frac{b^2}{a^2} = 1 - e^2$,

$$r^2(1 - e^2\cos^2\theta) = b^2;$$

$$\therefore\ r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

In this formula it is indifferent from which vertex the angle θ is measured; it shews that of all lines drawn from the center to the curve, the semi-axis major is the greatest when $\theta = 0$, and the semi-axis minor the least when $\theta = \frac{\pi}{2}$.

Tangent and Normal to the Ellipse.

121. To find the equation to the tangent of an ellipse at a given point.

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As in former cases, we regard the tangent as a secant which passes at first through two points of the curve, and then turns about one of them till the other moves up and coincides with it.

Let x', y' be the co-ordinates of the given point in the curve, and x'', y'' those of another point in the curve near the former, and let α' be the angle which the line joining them makes with the axis of x.

$$\text{then } \tan\alpha' = \frac{y' - y''}{x' - x''}.$$

But the two points being in the ellipse, their co-ordinates must satisfy its equation;

$$\therefore\ a^2 y'^2 + b^2 x'^2 = a^2 b^2,\quad a^2 y''^2 + b^2 x''^2 = a^2 b^2.$$

Subtracting the latter from the former we get

$$a^2(y'^2 - y''^2) + b^2(x'^2 - x''^2) = 0,$$

$$\text{which gives } \tan\alpha' = \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\cdot\frac{x' + x''}{y' + y''}.$$

Now let the second point move up to and coincide with the first, then $x'' = x'$, $y'' = y'$ and the secant becomes a tangent at (x', y'); therefore, denoting by α the angle which the tangent makes with the axis of x, we get

$$\tan\alpha = -\frac{b^2 x'}{a^2 y'};$$

therefore the equation to the tangent will be

$$y - y' = -\frac{b^2 x'}{a^2 y'}\cdot(x - x'),$$

or under another form, recollecting that $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$a^2 y y' + b^2 x x' = a^2 b^2.$$

122. The formula $m = \tan\alpha = -\frac{b^2}{a^2}\cdot\frac{x'}{y'}$, since it does not change when x' and y' are replaced by $-x'$ and $-y'$,

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shews that if PC be produced to meet the ellipse in P' (fig. 39) the tangents at P and P' are parallel, as we might have foreseen from the symmetrical position of the ellipse, relative to the axes; also it proves that at the points B, B', for which $x' = 0$, $y' = \pm b$, $\tan \alpha = 0$, or the tangents are parallel to AA'; and at A, A', for which $x' = \pm a$ and $y' = 0$, the tangents are perpendicular to AA'; and that for intermediate points, going from A to B, the angle PTx which is always obtuse, continually increases till PT becomes parallel to $A'A'$ at B.

123. To find where the tangent meets the axis major, put $y = 0$, then $x = \frac{a^2}{x'}$, or $CT = \frac{CA^2}{CN}$. As this result is independent of b, it will be the same for all ellipses constructed upon AA' as an axis; consequently if NP meet the circle whose diameter is AA' in Q, the tangent to the circle at Q will pass through T.

124. Subtracting CN or x' from the value just found for CT, we get the subtangent

$$NT = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

125. To find the equation to the normal at any point of an ellipse.

It will be of the form

$$y - y' = m'(x - x'),$$

and since it is perpendicular to the tangent,

$$m' = -\frac{1}{m} = \frac{a^2 y'}{b^2 x'}.$$

therefore the equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'} \cdot (x - x').$$

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126. To determine the point G, where the normal meets the axis major.

Make $y = 0$.

$$\therefore \quad x = x'\left(1 - \frac{b^2}{a^2}\right),$$

$$\text{or} \quad CG = e^2 . CN \quad \text{(fig. 41)},$$

$$\text{and the subnormal} \quad GN = \frac{b^2}{a^2} . CN.$$

127. The normal at any point bisects the angle between the focal distances of that point.

$$\text{For} \quad SG = ae + e^2x' = e(a + ex') = e . SP,$$

$$\text{and} \quad HG = ae - e^2x' = e(a - ex') = e . HP.$$

$$\therefore \quad \frac{SG}{HG} = \frac{SP}{HP},$$

and consequently the normal PG bisects the angle SPH (Euc. vi. 3). Also drawing the tangent YZ at P, since $GPY = GPZ$, each of them being a right angle, and $GPS = GPH$; therefore the angle $SPY = HPZ$, or the focal distances make equal angles with the tangent on the same side of it.

128. These properties furnish a simple method of drawing a tangent to an ellipse through a given point.

First let the point be in the ellipse as P (fig. 42); join SP, HP, and produce SP to K, making $SK = 2AC$; join HK and draw PZ perpendicular to it, then PZ is a tangent at P.

For $PK = 2AC - SP = PH$, PZ is common, and the angles at Z are right angles, therefore angle

$$HPZ = KPZ = SPY,$$

and consequently PZ is a tangent at P.

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129. Next let the point be without the ellipse.

With center S (fig. 43), and radius $= 2AC$ describe a circle KK', and with center T, the given external point, and radius TH, describe a circle cutting the former in K, K'; join SK meeting the ellipse in P, and join TP; then TP is a tangent at P; for $PK = PH$, $TK = TH$, and TP is common, therefore TP bisects the exterior angle HPK and is a tangent at P; similarly if SK' be joined meeting the the ellipse in P', a second point of contact will be found.

There is no difficulty in shewing that as long as T is exterior to the ellipse, the circles must intersect in two points.

130. The locus of the feet of the perpendiculars dropped from the foci upon the tangent to an ellipse, is the circumference of the circle whose diameter is the axis major.

For joining CZ (fig. 42), since SH is bisected in C and HK in Z, CZ is parallel to SK and equal to $\frac{1}{2}SK = AC$. Also drawing SY, CQ parallel to HZ, CQ bisects YZ perpendicularly, and therefore $CY = CZ = AC$.

131. Since C is the center of the circle which is the locus of Y and Z, and SYZ is a right angle, if YS and ZC be produced to meet in H', this will be a point in the circumference of the circle; and $SH' = HZ$.

hence $SY \times HZ = SY \cdot SH' = AS \times A'S = BC^2$ (Art. 109).

132. Since $SY \times HZ = BC^2$,

and $\dfrac{HZ}{HP} = \dfrac{SY}{SP}$, or $\dfrac{SY}{HZ} = \dfrac{SP}{HP}$ (Art. 127);

therefore multiplying together these equations

$$SY^2 = BC^2 \times \frac{SP}{HP};$$

or if SY be denoted by p and SP by r, between the radius vector and the perpendicular on the tangent from the focus we have the relation

$$p^2 = b^2 \cdot \frac{r}{2a - r}.$$

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133. Draw HI parallel to YZ, and suppose the angle $SPY = a$,

$$\text{then } \tan STP = \tan SHI = \frac{SI}{IH} = \frac{r \sin a - (2a - r) \sin a}{r \cos a + (2a - r) \cos a}$$

$$= \frac{2(r - a)}{2a} \tan a,$$

$$\text{or } \tan STP = \left(\frac{SP}{AC} - 1\right) \tan SPY;$$

a result which is sometimes of use.

134. Draw CE parallel to PY, then PC is a parallelogram,

$$\text{and } PE = CZ = AC.$$

135. If from G the foot of the normal at P (fig. 41), GL be drawn perpendicular to either focal distance, then $PL = \frac{1}{2}$ the latus rectum.

For by similar triangles, $\frac{SL}{SG} = \frac{SN}{SP}$;

$$\therefore \quad SL = SN \cdot \frac{SG}{SP} = e(ae + x) \quad \text{(Art. 127)};$$

$$\therefore \quad PL = a + ex - (ae^2 + ex) = a(1 - e^2) = \text{half the latus rectum.}$$

136. If a perpendicular be dropped from the center upon the tangent at any point of an ellipse, making an angle ϕ with the axis major, its length $= a\sqrt{1 - e^2 \sin^2 \phi}$.

Let CQ, HZ (fig. 44), be perpendiculars upon PT the tangent at any point P from the center and focus; join CZ and let $TCQ = \phi$,

$$\text{then } CZ = a, \text{ and } QZ = CH \sin \phi = ae \sin \phi.$$

$$\therefore \quad CQ^2 = a^2 - a^2 e^2 \sin^2 \phi,$$

$$\text{or } CQ = a\sqrt{1 - e^2 \sin^2 \phi}.$$

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137. Let $P'T'$ (fig. 44) be another tangent to the ellipse at right angles to the former, and CQ' perpendicular to it, then $Q'CT' = \frac{1}{2}\pi - \phi$;

and therefore $CQ'^2 = a^2 - a^2e^2\cos^2\phi$.

Hence $CW^2 = CQ^2 + CQ'^2 = a^2 + a^2(1 - e^2) = a^2 + b^2$,

and therefore the locus of the intersection of two tangents to an ellipse at right angles to one another is a circle whose center is C and radius equal to $\sqrt{a^2 + b^2}$.

The Ellipse referred to its conjugate Diameters.

138. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CD (fig. 45) through the center, whose equation is $y = mx$; then the equation to any one of them QQ' will be

$$y = mx + c,$$

and to determine the points in which it meets the ellipse, we must combine its equation with the equation to the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2,$$

this gives, eliminating x by the substitution $\frac{1}{m}(y - c)$,

$$y^2 - \frac{2b^2c}{a^2m^2 + b^2}y + \frac{b^2(c^2 - m^2a^2)}{a^2m^2 + b^2} = 0, \quad 1$$

the roots of which are represented by QM, $Q'M'$; but if V be the middle point QQ', and $CN = X$, $VN = Y$ its coordinates,

$$Y = \tfrac{1}{2}(QM + Q'M') = \frac{b^2c}{m^2a^2 + b^2},$$

$$X = \frac{1}{m}(Y - c) = -\frac{ma^2c}{m^2a^2 + b^2};$$

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therefore, dividing one result by the other in order to eliminate the quantity c, which particularizes the chord, we get

$$Y = -\frac{b^2}{ma^2} \cdot X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line PP' passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a diameter. Hence all diameters of an ellipse pass through its center; and, conversely, every line through the center is a diameter.

139. Hence denoting the equation to any chord by

$$y = mx + c,$$

and the equation to the diameter which bisects it by $y = m'x$, we have

$$m' = -\frac{b^2}{ma^2}, \quad \text{or} \quad mm' = -\frac{b^2}{a^2},$$

a simple relation by means of which the equation of one may always be deduced from that of the other.

140. The chords which are parallel to PP' are bisected by CD.

For any one of these chords RR' may be represented by the equation

$$y = m'x + c',$$

then the diameter which bisects them, by what has been proved, will have for its equation

$$y = -\frac{b^2}{m'a^2} \cdot x, \quad \text{or} \quad y = mx, \quad \text{since} \quad mm' = -\frac{a^2}{b^2},$$

which belongs to CD.

Hence two diameters, whose equations $y = mx$, $y = m'x$ are so related, that $mm' = -\frac{b^2}{a^2}$, have the property that

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each bisects the chords parallel to the other; they are called conjugate diameters, or rather those portions of them PP', DD', which fall within the ellipse are usually called a pair of conjugate diameters.

141. If PT be the tangent at P, and x', y', the co-ordinates of P, then $m = \frac{y'}{x'}$, therefore the equation to PT is

$$y - y' = -\frac{b^2}{m a^2} \cdot (x - x')$$

which represents a line parallel to CD.

Hence the tangent applied at the extremity of any diameter is parallel to the corresponding conjugate diameter; and if tangents be drawn at the extremities of a pair of conjugate diameters, they will form a parallelogram circumscribing the ellipse, (Art. 122.)

142. Having given the co-ordinates of the extremity of any diameter, to find those of the extremity of the diameter conjugate to it.

Let x', y', be the co-ordinates of the point P (fig. 46),

then $y = \frac{y'}{x'} x$ is the equation to CP,

and $\therefore y = -\frac{b^2 x'}{a^2 y'} x$ the equation to CD.

To determine the co-ordinates of D, we must combine the equation to CD with the equation to the ellipse; which gives, eliminating y by the substitution $-\frac{b^2 x'}{a^2 y'} x$,

$$\frac{b^4}{a^2} \cdot \frac{x'^2}{y'^2} \cdot x^2 + b^2 x^2 = a^2 b^2,$$

$$\text{or } x^2 (b^2 x'^2 + a^2 y'^2) = a^4 y'^2,$$

$$\text{or } x^2 = \frac{a^2 y'^2}{b^2};$$

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$$\therefore\ x = CM = -\frac{ay'}{b},$$

$$\text{and } y = DM = \frac{bx'}{a}.$$

143. The sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.

$$CP^2 = x'^2 + y'^2,$$

$$CD^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = a^2 - x'^2 + b^2 - y'^2, \text{ since } y'^2 = \frac{b^2}{a^2}(a^2 - x'^2);$$

$$\therefore\ CP^2 + CD^2 = a^2 + b^2.$$

144. All parallelograms whose sides touch an ellipse at the extremities of a pair of conjugate diameters, are equal to one another.

Draw PF (fig. 46) perpendicular to DC produced.

Then area of whole parallelogram

$$= 4CD \cdot PF$$

$$= 4CD \cdot CT \cdot \sin TCF = 4CT \cdot DM$$

$$= 4 \cdot \frac{a^2}{x'} \cdot \frac{bx'}{a} = 4ab \text{ (Art. 123)}.$$

145. If we denote CP, CD by a', b', and the angle DCP by γ, we have

$PF = a' \sin \gamma$, and therefore $a'b' \sin \gamma = CD \cdot PF = ab$.

146. Also if we denote CQ or PF by p, we have

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a^2 + b^2 - a'^2},$$

a relation between the central distance of a point, and the perpendicular upon the tangent at that point, let fall from the center.

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147. The rectangle contained by the focal distances of any point is equal to the square of the corresponding semi-conjugate diameter.

$$CD^2 = a^2 + b^2 - CP^2 \text{ (fig. 40)}$$

$$= a^2 + b^2 - x^2 - b^2 + \frac{b^2}{a^2} x^2$$

$$= a^2 - e^2 x^2 = (a + ex) \cdot (a - ex)$$

$$= SP \cdot HP \text{ (Art. 115).}$$

148. To find the equation to the ellipse referred to any pair of conjugate diameters as axes.

The equation to the ellipse referred to its center and axes is

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

Let the conjugate diameters CP, CD (fig. 47), be the new axes of x' and y', inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes will be (Art. 42),

$$x = x' \cos \alpha + y' \cos \beta, \quad y = x' \sin \alpha + y' \sin \beta.$$

Hence, substituting and reducing,

$$(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)\, x'^2 + (a^2 \sin^2 \beta + b^2 \cos^2 \beta)\, y'^2$$

$$+ 2\,(a^2 \sin \alpha \sin \beta + b^2 \cos \alpha \cos \beta)\, x'y' = a^2 b^2.$$

$$\text{But } \tan \alpha \tan \beta = -\frac{b^2}{a^2} \text{ (Art. 140);}$$

$$\therefore\ a^2 \sin \alpha \sin \beta + b^2 \cos \alpha \cos \beta = 0.$$

Also if $CP = a'$, $CD = b'$, we have (Art. 120),

$$a'^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) = a^2 b^2,$$

$$b'^2 (a^2 \sin^2 \beta + b^2 \cos^2 \beta) = a^2 b^2;$$

therefore, substituting and dividing by $a^2 b^2$, we get for the required equation,

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$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

or, in a geometrical form, supposing PC produced to meet the ellipse in G,

$$QV^2 = \frac{CD^2}{CP^2} . PV . VG.$$

149. This equation, which, suppressing the accents of the variables, is $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, being precisely of the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the axes of the ellipse and to its conjugate diameters.

Hence, x, y being the co-ordinates of any point Q referred to the conjugate diameters CP, CD (fig. 47), the equation to the tangent at that point will be

$$a'^2yy' + b'^2xx' = a'^2b'^2,$$

and if it meet the axis of x in T, we shall have

$$CT = \frac{CP^2}{CV} \text{ as before.}$$

150. Also if we wish to draw a tangent through an external point Q (fig. 48), whose co-ordinates are h and k, we shall have, to find the points of contact, the equations

$$a'^2y'^2 + b'^2x'^2 = a'^2b'^2,$$

$$a'^2ky' + b'^2hx' = a'^2b'^2.$$

And those points, as in preceding similar cases, may be determined by constructing the line whose equation is

$$a'^2ky + b'^2hx = a'^2b'^2,$$

that is, by taking $CT = \frac{a'^2}{h}$, $CR = \frac{b'^2}{k}$, and joining RT which will cut the ellipse in the two required points.

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Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any other point in this line we draw a pair of tangents to the ellipse, the secant passing through the new points of contact, will cut the diameter CP in T, as this point can only alter when h alters. Hence if from the several points of any straight line pairs of tangents be drawn to an ellipse, the straight lines which join the corresponding points of contact will all pass through the same point.

151. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$b'^2xx' \pm a'^2yy' = a'^2b'^2,$$

where the upper or lower sign is to be used according as we consider the point Q (x', y'), or the other extremity of the chord Q' whose co-ordinates are $x', -y'$; and in both cases when $y = 0$, $x = \frac{a'^2}{x'}$, therefore the tangents meet the axis of x in the same point T, (fig. 47).

152. If from any point within or without an ellipse, two lines be drawn parallel to two given straight lines to meet the curve, the rectangle of the segments will be to one another in an invariable ratio.

Let O (fig. 49) be the given point with co-ordinates h and k. Then the polar equation to the ellipse, reckoning from O, will be (Art. 44)

$$a^2(r\sin\theta + k)^2 + b^2(r\cos\theta + h)^2 = a^2b^2,$$

which is of the form $r^2 + Mr - N = 0$,

where $N = \frac{a^2b^2 - a^2k^2 - b^2h^2}{a^2\sin^2\theta + b^2\cos^2\theta} = r'r''$, if these be the two values of r.

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Now let Pp, Qq be drawn parallel to CP', CQ', which make given angles α, β with Cx, then

$$PO \times Op : QO \times Oq :: a^2 \sin^2\beta + b^2 \cos^2\beta : a^2 \sin^2\alpha + b^2 \cos^2\alpha$$
$$:: CP'^2 : CQ'^2 \quad \text{(Art. 120)},$$

a ratio independent of the position of the point O.

153. To find the area of the ellipse.

Let $APQRTA'$ (fig. 50) be any polygon inscribed in the ellipse, and let the ordinates PN, QM, &c. be produced to meet the circle on the major axis in p, q, r, &c. and join Ap, pq, &c.

Then area of trapezium $PNMQ = \frac{1}{2}(PN + QM) \cdot NM$

$$= \frac{1}{2}\frac{b}{a}(pN + qM) \cdot NM$$

$$= \frac{b}{a} \cdot \text{trapezium } pNMq,$$

$$\text{or} \quad \frac{\text{area of trapezium } PM}{\text{area of trapezium } pM} = \frac{b}{a},$$

and since the same ratio exists between every two corresponding trapeziums,

$$\frac{\text{area of polygon } APQA'}{\text{area of polygon } ApqA'} = \frac{b}{a},$$

and this is true however much the number of the sides of the polygons be increased; therefore supposing the number to be infinite, in which case the ratio of the polygons becomes that of the semi-ellipse and semicircle,

$$\frac{\text{area of semi-ellipse}}{\text{area of semicircle}} = \frac{b}{a};$$

$$\therefore \text{ area of ellipse} = \frac{b}{a}\pi a^2 = \pi ab.$$

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154. Let K (fig. 51) be any point in the axis, and QPN an ordinate to the circle and ellipse, then

$$\text{elliptic area } ANP = \frac{b}{a} . \text{ circular area } ANQ,$$

$$\text{and triangle } PKN = \frac{b}{a} . \text{ triangle } QKN;$$

therefore, subtracting,

$$\text{the elliptic sectorial area } AKP = \frac{b}{a} . \text{ circular area } AKQ.$$

Also if a', b' be two semi-conjugate diameters and γ the angle between them, the area of the ellipse $= \pi a' b' \sin \gamma$ (Art. 145).

And the area of the sector bounded by the semi-diameters

$$= \tfrac{1}{4} \pi a' b' \sin \gamma.$$

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SECTION VIII.

ON THE HYPERBOLA.

Various forms of the equation to the Hyperbola.

155. To find the equation to the Hyperbola.

The hyperbola is the locus of a point whose distance from a given point is always greater than its distance from a given fixed line in a constant ratio.

directrix KK'.

Let KK' (fig. 52) be the given fixed line, and S the given point, from which draw SX perpendicular to KK'. Let P be a point in the hyperbola on either side of KK', and from P draw PM perpendicular to KK' and join SP, and let the constant ratio of SP to PM be $e : 1$, e being greater than 1. Divide the given distance SX in A so that $SA = e \cdot AX$, then A is a point in the curve, and assuming $AS = p$, we have $AX = \frac{p}{e}$. Through A draw Ay parallel to KK' and take A for the origin and Ax, Ay, for the co-ordinate axes, and let $AN = x$, $NP = y$ be the co-ordinates of P; then

$$SP^2 = e^2 \cdot PM^2,$$

$$\text{or } SN^2 + NP^2 = e^2 \cdot NX^2,$$

$$\text{or } (x-p)^2 + y^2 = e^2 \cdot \left(\frac{p}{e} + x\right)^2 = (p + ex)^2;$$

$$\therefore y^2 = 2p(e+1)x + (e^2-1)x^2,$$

$$\text{or } y^2 = (e^2-1)\left(\frac{2p}{e-1}x + x^2\right),$$

or, if we replace the known quantity $\frac{p}{e-1}$ by a,

$$y^2 = (e^2-1)(2ax + x^2),$$

the required equation.

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156. To determine the points where the curve cuts the axis of x, make $y = 0$, then $x = 0$, or $x = -2a$; the value $x = 0$ gives the point A already known, the other value $x = -2a = AA'$, determines the point A'. Bisect AA' in C, then in the equation to the hyperbola making $x = -a = AC$, we get $y^2 = -(e^2 - 1)a^2$ or $y = \pm a\sqrt{e^2 - 1} \cdot \sqrt{-1}$, which are imaginary values; hence the curve does not, as in the case of the ellipse, meet the line BB' drawn perpendicular to AA' through its middle point; if however we put $a\sqrt{e^2 - 1} = b$ and take BC, $B'C$, each equal to b, BB' will be denoted by $2b$, and the equation will become

$$y = \pm \frac{b}{a}\sqrt{2ax + x^2}.$$

157. In order to transfer the origin to C, we must change x into $x' - a$, since $AN = CN - CA$;

$$\therefore\ y^2 = \frac{b^2}{a^2}\{2a(x' - a) + (x' - a)^2\} = \frac{b^2}{a^2}(x'^2 - a^2).$$

This form of the equation shews that the origin is the center of the hyperbola, and that the co-ordinate axes are axes of the hyperbola (Art. 64); but the term axis is more particularly appropriated to the portions of those lines, $AA' = 2a$, $BB' = 2b$; the former of which meets the hyperbola and is called the transverse axis, and its extremities are called the vertices of the hyperbola; and the latter, although the line along which it is measured does not meet the curve, is taken for the second axis of the hyperbola and is called the conjugate axis. Since $b = a\sqrt{e^2 - 1}$ where e is only restricted to be greater than 1, b may be either greater or less than a.

158. To trace the hyperbola by means of its equation.

The equation to the hyperbola referred to its axes is

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

hence for all values of x between $+a$ and $-a$, y is imaginary, and therefore no part of the curve lies in the space

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bounded by the two indefinite lines through A, A', parallel to BC. When $x = a$, $y = 0$, and as x increases positively from a to ∞, the two values of y are real and increase from zero to ∞, and give the infinite branch ZAx situated symmetrically with respect to Cx; and since when the sign of x is changed the values of y do not alter, the negative values of x will give a branch $Z'A'x'$ precisely similar to the former on the other side of BB', which is described by taking $SP' : P'M$ as $e : 1$. Moreover the two opposite branches of which the hyperbola is composed will everywhere turn their convexities towards the axis BB', otherwise a straight line might intersect them in more than two points, which is impossible.

159. Since the hyperbola is symmetrically situated with respect to its axes, if we take $CH = CS$, $CX' = CX$, and draw kX' parallel to CB, the curve may be described by means of the focus H and directrix kX', exactly in the same way as by means of S and KX. Hence the hyperbola has two foci situated in the transverse axis at equal distances from its center. Also since

$$a = \frac{p}{e-1} \text{ we have } AS = a(e-1).$$

$$\therefore \quad SC = a + a(e-1) = ae, \quad \text{and} \quad e = \frac{SC}{AC}.$$

160. The quantity e which expresses the ratio of the distance between either focus and the center to the semi-transverse axis is called the eccentricity. Since $b = a\sqrt{e^2-1}$, the eccentricity expressed in terms of the semi-axes is equal to $\frac{\sqrt{a^2+b^2}}{a}$. Hence $SC = \sqrt{a^2+b^2}$, and $AS \cdot AH = b^2$.

161. Since $AS = e \cdot AX$, we have $AX = \frac{a(e-1)}{e}$.

$$\therefore \quad CX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS}.$$

which determines the directrix relative to the center.

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162. The double ordinate passing through the focus is called the latus rectum. To find its value make $x = CS = ae$ (fig. 53).

$$\therefore \quad y^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2 - 1) = \frac{b^4}{a^2};$$

$$y = \pm\frac{b^2}{a}, \quad \therefore \quad LL' = \frac{2b^2}{a} = 2a(e^2 - 1).$$

163. In the hyperbola when the distance CS between the focus and center becomes infinite, the distance AS between the focus and vertex remaining finite, the curve is changed into a parabola.

The equation is $y^2 = 2a(e^2 - 1)x + (e^2 - 1)x^2$,

or $y^2 = 2p(e + 1)x + (e^2 - 1)x^2$ if $AS = p$.

But $e = \frac{SC}{AC} = \frac{SC}{SC - p} = \frac{1}{1 - \frac{p}{SC}}$; let $SC = \infty$, $\therefore e = 1$;

$\therefore \quad y^2 = 4px$, the equation to a parabola.

164 When $b = a$, the above equations become

$$y^2 = x^2 - a^2, \quad y^2 = 2ax + x^2.$$

The hyperbola in this case is called rectangular, and it is to the ordinary hyperbola what the circle is to the ellipse.

165. Since $y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}(x + a)(x - a)$,

gives $PN^2 = \frac{BC^2}{AC^2} \cdot A'N \cdot AN$,

we see that the square of the ordinate varies as the rectangle of the distances of its foot from the extremities of the transverse axis.

166. The equation to the hyperbola results from that to the ellipse by changing b^2 into $-b^2$, or b into $b\sqrt{-1}$.

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This remark may be of use in enabling us to foresee those properties of the hyperbola which are analogous to properties of the ellipse.

167. To express the distances of any point in the hyperbola from the foci in terms of its abscissa.

$$SP^2 = SN^2 + NP^2 \text{ (fig. 53)}$$
$$= (x - ae)^2 + (e^2 - 1)(x^2 - a^2)$$
$$= (ex - a)^2;$$

$\therefore SP = \pm (ex - a)$; now as long as P is in that branch of the hyperbola of which S is the interior focus, ex is greater than a; therefore in order that SP may be positive we must take the upper sign, which gives

$$SP = ex - a.$$

Also $HP^2 = (x + ae)^2 + (e^2 - 1)(x^2 - a^2) = (ex + a)^2.$

$$\therefore \quad HP = ex + a.$$

168. Exactly in the same way as for the ellipse, it may be shewn that the foci are the only points whose distances from any point in the curve can be expressed rationally in terms of the abscissa of the point. (Art. 116).

169. Hence, subtracting,

$$HP - SP = 2a,$$

or the difference of the focal distances of any point in the hyperbola is constant, and equal to the transverse axis. This property affords a simple method of determining any number of points in a hyperbola of which we know the transverse axis and foci. In $A'A$ (fig. 53) produced take any point F and with centers S and H and radii respectively equal to AF, $A'F$, describe circles intersecting in P, P'; these are manifestly points in the hyperbola. The curve may be described by a continuous motion if we have a rule HM (fig. 53) moveable about the focus H, and a

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string SPM fastened to M and to the other focus S, of such a length that $HM - SPM = AA'$; then as HM revolves about H, if a point P slide along HM so as always to confine a portion of string PM against it, the point will trace out a hyperbola.

This property also furnishes the following method of investigating the equation to the hyperbola.

170. To find the locus of a point the difference of whose distances from two fixed points is constant.

Through the two fixed points S, H (fig. 53) draw the indefinite line Hx, bisect SH in C and through C draw Cy perpendicular to it, and take Cx, Cy for the axes of the co-ordinates, as the locus will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point, and $HP - SP = 2a$.

$$\text{Then } HP^2 = (x + c)^2 + y^2,$$

$$SP^2 = (x - c)^2 + y^2;$$

$$\therefore HP^2 - SP^2 \quad \text{or } 2a(HP + SP) = 4cx;$$

$$\therefore HP + SP = \frac{2cx}{a};$$

$$\text{but } HP - SP = 2a;$$

$$\therefore HP = \frac{cx}{a} + a;$$

$$\therefore \left(\frac{cx}{a} + a\right)^2 = (x + c)^2 + y^2,$$

$$\text{or } y^2 = \frac{c^2 - a^2}{a^2}(x^2 - a^2).$$

Now $HP - SP$ is less than SH or $a < c$, consequently $c^2 - a^2$ is a positive quantity, and the equation, as we should expect, represents a hyperbola whose transverse axis is $2a$ and conjugate axis $2\sqrt{c^2 - a^2}$.

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171. To find the polar equation to the hyperbola, one of the foci being the pole.

Let the interior focus be the pole, $SP = r$, $xSP = \theta$ (fig. 53), then $SN = r\cos\theta$ and $CN = ae + r\cos\theta$;

but $SP = e \cdot CN - a$;

$$\therefore\ r = ae^2 + er\cos\theta - a,$$

$$\text{or } r = \frac{a(e^2-1)}{1 - e\cos\theta}.$$

Since $e > 1$, there is some angle $xSD = \alpha$ whose cosine $= \frac{1}{e}$; for values of θ less than α, r is negative and there are no points in the branch AZ corresponding to those values, because $SP = ex - a$ is always positive. When $\theta = \alpha$, r is infinite and the radius vector meets the curve at an infinite distance; when θ exceeds α, r is positive, and as θ increases to π we get the portion of the curve ZA; when θ increases beyond π, the same values of r recur in an inverse order, giving the portion Ax till $\theta = 2\pi - \alpha$, when r is again infinite and afterwards becomes negative.

172. If S be the exterior focus and $SP' = r$, $xSP' = \theta$, $SP' = e \cdot CN' + a$; and $CN' = SP'\cos P'SN' - ae = -r\cos\theta - ae$;

$$\therefore\ r = -re\cos\theta - ae^2 + a,$$

$$\text{or } r = \frac{-a(e^2-1)}{1 + e\cos\theta}.$$

In this case also, r is negative till $\theta = \pi - \alpha$, when it becomes infinite, and then produces the branch of the hyperbola $Z'A'x'$ as θ changes from $\pi - \alpha$ to $\pi + \alpha$.

There is no difficulty in shewing that if we remove the restriction of having r positive, and measure negative values upon the radius vector produced backwards, the same equation will represent both branches of the hyperbola.

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173. To find the polar equation to the hyperbola, the center being the pole.

Let $CP = r$, $xCP = \theta$ (fig. 53), then $x = r\cos\theta$, $y = r\sin\theta$, and substituting in the equation $a^2y^2 - b^2x^2 = -a^2b^2$, we get

$$r^2(a^2\sin^2\theta - b^2\cos^2\theta) = -a^2b^2,$$

or dividing by a^2 and observing that $\frac{b^2}{a^2} = e^2 - 1$,

$$r^2(1 - e^2\cos^2\theta) = -b^2;$$

$$\therefore\ r = \frac{b}{\sqrt{e^2\cos^2\theta - 1}}.$$

Taking θ from zero to α, where $\cos\alpha = \frac{1}{e}$, we get the part AZ; from $\theta = \alpha$ to $\theta = \pi - \alpha$, r is imaginary; from $\theta = \pi - \alpha$ to $\theta = \pi + \alpha$ we get the branch $Z'A'x'$; and the remaining portion Ax in taking θ from $2\pi - \alpha$ to 2π.

Tangent and Normal to the Hyperbola.

174. To find the equation to the tangent of a hyperbola at a given point.

The equation to the hyperbola being $a^2y^2 - b^2x^2 = -a^2b^2$, in order to find the equation to the tangent the process will be the same as for the ellipse, with the sole difference that b^2 will every where be replaced by $-b^2$, and the result will be the same, subject to that modification. Hence if x', y', be the co-ordinates of the point of contact and α the angle which the touching line makes with the axis of x,

$$\tan\alpha = \frac{b^2x'}{a^2y'},\quad \text{(Art. 121)}$$

and the equation to the tangent will be

$$a^2yy' - b^2xx' = -a^2b^2.$$

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175. The formula $\tan \alpha = \frac{b^2}{a^2} \frac{x'}{y'}$,

since it does not alter when x' and y' are replaced by $-x'$ and $-y'$, shews that if PC (fig. 54) be produced to meet the hyperbola in P', the tangents at P and P' are parallel as we might have foreseen on account of the symmetrical position of the hyperbola relative to its axes; and that at the points A, A' for which $y' = 0$, $x' = \pm a$, $\tan \alpha = \infty$, or the tangents are perpendicular to the axis. Also replacing x' by its value

$$\frac{a}{b}\sqrt{y'^2 + b^2}, \text{ we get } \tan \alpha = \frac{b}{a}\sqrt{1 + \frac{b^2}{y'^2}},$$

which shews that as y' increases from zero to ∞, $\tan \alpha$ diminishes from ∞ to a limit $\frac{b}{a}$.

176. To find where the tangent meets the transverse axis, make in the equation to the tangent $y = 0$;

$$\therefore x = \frac{a^2}{x'}, \text{ or } CT = \frac{CA^2}{CN}.$$

As this result is independent of b, it will be the same for all hyperbolas described with the same transverse axis. The value of CT diminishes as x' increases, but is always of the same sign, and becomes zero when $x' = \infty$. Hence when the point of contact is at an infinite distance, the tangent passes through the center and makes with the axis of x an angle whose tangent is $\frac{b}{a}$; i. e. it coincides with the diagonal CW of the rectangle constructed with the semi-axes. This line is called an asymptote to the hyperbola.

177. The subtangent $NT = CN - CT = x' - \frac{a^2}{x'} = \frac{x'^2 - a'^2}{x'}$.

178. To find the equation to the normal of a hyperbola at a given point.

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Let x', y', be the co-ordinates of the given point, then since the normal is perpendicular to the tangent at that point, it is easily seen that its equation is

$$y - y' = -\frac{a^2 y'}{b^2 x'}(x - x').$$

179. To determine the point G (fig. 54) where the normal meets the transverse axis; make in the equation to the normal

$$y = 0; \quad \therefore \ x = x'\left(1 + \frac{b^2}{a^2}\right), \text{ or } CG = e^2 . CN.$$

Hence, the least value of CG, when $x' = a$, is $\frac{a^2 + b^2}{a}$.

The subnormal $NG = CG - CN = \frac{b^2}{a^2} . x' = \frac{b^2}{a^2} . CN.$

180. The normal at any point bisects the exterior angle formed by the focal distances of that point.

For $SG = CG - CS = e^2 x' - ae = e . SP,$

$$HG = CG + CH = e^2 x' + ae = e . HP;$$

$$\therefore \frac{SG}{HG} = \frac{SP}{HP}; \quad \therefore PG \text{ bisects the angle } SPh. \text{ (Euc. VI. Prop. A.)}$$

Also since GPT, GPt, are right angles, and $SPG = hPG$;

$$\therefore SPT = hPt = HPT,$$

or the focal distances make equal angles with the tangent on opposite sides of it.

181. These properties furnish a simple method of drawing a tangent to the hyperbola through a given point.

First, let the point be in the curve as P (fig. 55), join SP, HP; make $HK = 2AC$, join SK and draw PY perpendicular to it, then $PK = HP - 2AC = SP$, PY is common, and the angles at Y right angles; $\therefore \angle SPY = KPY$, and consequently PY is a tangent at P.

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182. Next let the point be without the hyperbola, as T (fig. 56).

With center H and radius $= 2AC$ describe a circle KK', and with center T the given external point and radius TS describe a circle cutting the former in K, K', join HK meeting the hyperbola in P, and join TP, then TP is a tangent at P.

for $SP = HP - 2AC = PK$, $TS = TK$, and PT is common,

$\therefore$ PT bisects the angle SPH and is therefore a tangent at P.

Similarly, if HK' be joined and produced to meet the hyperbola in P', a second point of contact will be determined.

183. The locus of the feet of the perpendiculars dropped from the foci upon the tangent to a hyperbola is the circumference of the circle whose diameter is the transverse axis.

For joining CY (fig. 55), since SH is bisected in C, and SK in Y, CY is parallel to HK and $= \frac{1}{2}HK = AC$. Also drawing HZ, CQ perpendicular to ZY, Q is the middle point of ZY, and therefore $CZ = CY = CA$.

184. Since C is the center of the circle which is the locus of Y and Z, and SYZ is a right angle, if SY and ZC be produced to meet in S', this will be a point in the circumference and $SS' = HZ$.

$\therefore$ $SY \times HZ = SY \times SS' = SA \times SA' = BC^2$. (Art. 160)

185. Since $SY \times HZ = BC^2$, and $\frac{SY}{SP} = \frac{HZ}{HP}$ or $\frac{SY}{HZ} = \frac{SP}{HP}$,

$$\therefore \quad SY^2 = BC^2 \times \frac{SP}{HP};$$

or if SP, SY, be denoted respectively by r and p, and consequently $HP = 2a + r$, we have

$$p^2 = b^2 \cdot \frac{r}{2a + r}.$$

186. Draw CE parallel to PY (fig. 55), then CP is a parallelogram, and $PE = CY = CA$.

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The Hyperbola referred to its Conjugate Diameters.

187. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CW through the center (fig. 57), whose equation is $y = mx$; then the equation to any one of the chords QQ' will be $y = mx + c$, and to determine the points in which it meets the hyperbola, we must combine its equation with that to the hyperbola.

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

This gives, eliminating x by the substitution $\frac{1}{m}(y - c)$,

$$y^2 + \frac{2b^2c}{m^2a^2 - b^2} \cdot y + b^2 \cdot \frac{m^2a^2 - c^2}{m^2a^2 - b^2} = 0,$$

the roots of which are represented by QM, $Q'M'$; but if V be the middle point of QQ' and $CN = X$, $NV = Y$, its co-ordinates, $2NV = QM + Q'M'$,

$$\therefore \quad Y = \frac{-b^2c}{m^2a^2 - b^2},$$

$$\text{and} \quad X = \frac{1}{m}(Y - c) = -\frac{ma^2c}{m^2a^2 - b^2}.$$

Dividing one result by the other in order to eliminate the quantity c which particularizes the chord, we get

$$Y = \frac{b^2}{ma^2}X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line CV passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a diameter; hence all diameters of a hyperbola pass through its center; and, conversely, every line through the center may be considered as a diameter.

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188. Hence, denoting the equation to any chord by $y = mx + c$, and the equation to the diameter which bisects it by $y = m'x$, we have

$$m' = \frac{b^2}{ma^2}, \text{ or } mm' = \frac{b^2}{a^2},$$

a simple relation by means of which the equation of one may be deduced from that of the other.

189. The chords which are parallel to CV are bisected by CW.

For any one of these chords RR' may be represented by the equation

$$y = m'x + c',$$

then the diameter which bisects them will have for its equation

$$y = \frac{b^2}{m'a^2}x, \text{ or } y = mx,$$

which belongs to CW. Hence two diameters, whose equations $y = mx$, $y = m'x$ are so related that $mm' = \frac{b^2}{a^2}$, have the property that each bisects the chords parallel to the other; they are called conjugate diameters. But the term is usually restricted to those portions of them PP', DD', which are intercepted by the proposed hyperbola, and the hyperbola which is conjugate to it (fig. 58).

190. The latter is a hyperbola $BDB'D'$ whose transverse and conjugate axes coincide respectively with the conjugate and transverse axes of the proposed curve; and the employment of it is attended with great conveniences in stating and investigating the properties of the hyperbola.

The equation to the conjugate hyperbola, referred to the same axis of x and axis of y as the primitive hyperbola so that $DM = y$, $CM = x$, (fig. 58) will consequently be

$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \text{ or } y^2 = \frac{b^2}{a^2}(x^2 + a^2),$$

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which we observe results from the equation to the primitive hyperbola by replacing a^2 and b^2 by $-a^2$ and $-b^2$.

191. If PT be a tangent at P (fig. 58), and x', y', the co-ordinates of P, then

$$m = \frac{y'}{x'}, \text{ and the equation to } PT \text{ is}$$

$$y - y' = \frac{b^2}{ma^2}(x - x')$$

which represents a line parallel to CD. Hence the tangent applied at the extremity of any diameter is parallel to the corresponding conjugate diameter. Similarly, the tangent to the conjugate hyperbola at D is parallel to CP; and if tangents be applied to the hyperbola and its conjugate, at the extremities of a pair of conjugate diameters, they will form a parallelogram.

192. Of any two conjugate diameters, only one can meet the hyperbola.

Let $y = mx$ be the equation to a diameter; to determine its intersection with the curve, put mx for y in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

and we find for the abscissæ of the points of intersection

$$x = \pm \sqrt{\frac{a^2b^2}{b^2 - m^2a^2}},$$

which are real as long as m is less than $\frac{b}{a}$, but imaginary if m be greater than $\frac{b}{a}$; in the former case the diameter intersects the curve, in the latter it does not. But the relation $mm' = \frac{b^2}{a^2}$ shews that if m be less than $\frac{b}{a}$, m' is greater than $\frac{b}{a}$; hence every diameter which meets the hyperbola, has its conjugate diameter amongst those which do not meet it.

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193. If we construct on the axes of the curve the rectangle Ll' (fig. 59), all the diameters which fall within the angle LCl, make with AC an angle whose tangent (abstracting the sign) is less than $\frac{b}{a}$; whilst the diameters which fall within the angle LCL' make with AC an angle whose tangent exceeds $\frac{b}{a}$; the former are those that meet the curve, the latter those which do not. In the particular case when $m = \frac{b}{a}$, we have also $m' = \frac{b}{a}$, and the conjugate diameters coincide with Ll', and meet the curve only at an infinite distance; similarly, when $m = -\frac{b}{a}$, we have $m' = -\frac{b}{a}$, and the two diameters coincide with the other diagonal $L'l$.

194. Having given the co-ordinates of the extremity of any diameter, to find those of the diameter conjugate to it.

Let CD be conjugate to CP (fig. 58), and let it meet the conjugate hyperbola in D; let x', y' be co-ordinates of P, and consequently $y = \frac{y'}{x'}x$ the equation to CP, then $y = \frac{b^2}{a^2} \cdot \frac{x'}{y'}x$ is the equation to CD; and to determine the co-ordinates of D we must combine this equation with the equation to the conjugate hyperbola, which is

$$a^2y^2 - b^2x^2 = a^2b^2.$$

This gives, eliminating y by the substitution of $\frac{b^2x'}{a^2y'}x$,

$$\frac{b^4}{a^2} \cdot \frac{x'^2}{y'^2}x^2 - b^2x^2 = a^2b^2,$$

$$\text{or } x^2(b^2x'^2 - a^2y'^2) = a^4y'^2,$$

$$\text{or } x^2 = \frac{a^2y'^2}{b^2};$$

$$\therefore x = CM = \frac{ay'}{b}, \quad \text{and } y = DM = \frac{bx'}{a}.$$

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195. The difference of the squares of any two semi-conjugate diameters, is equal to the difference of the squares of the semi-axes.

$$CP^2 = x'^2 + y'^2,$$

$$CD^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = x'^2 - a^2 + y'^2 + b^2, \text{ because } y'^2 = \frac{b^2}{a^2}(x'^2 - a^2);$$

$$\therefore CP^2 - CD^2 = a^2 - b^2.$$

196. All parallelograms whose sides touch a hyperbola and the conjugate hyperbola at the extremities of a pair of conjugate diameters, are equal to one another.

Draw PF perpendicular to CD (fig. 58), then

$$\text{area of whole parallelogram} = 4CD.PF = 4CD.CT\sin TCF$$

$$= 4DM.CT = 4\frac{bx'}{a}.\frac{a^2}{x'} = 4ab.$$

197. If we denote CP, CD by a', b', and angle PCD by γ, we have

$$PF = a'\sin\gamma, \quad \text{and} \quad \therefore a'b'\sin\gamma = CD \times PF = ab.$$

Also if we denote PF by p, we have the relation between the central distance of any point and the perpendicular from the center upon the tangent at that point,

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a'^2 - a^2 + b^2}.$$

198. Draw the diagonal CL (fig. 58), which will pass through the middle point of DP, whose co-ordinates are equal to $\frac{1}{2}(CM + CN)$, $\frac{1}{2}(DM + PN)$;

$$\therefore \tan LCA = \frac{\frac{1}{2}\left(y' + \frac{bx'}{a}\right)}{\frac{1}{2}\left(x' + \frac{ay'}{b}\right)} = \frac{b}{a};$$

hence the diagonals of all the parallelograms coincide with the limiting positions of the tangents (Art. 176); and the parallelo-

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grams are not only equal in area, but they all have their diagonals in the same straight lines.

199. The rectangle contained by the focal distances of any point, is equal to the square of the corresponding semi-conjugate diameter.

$$CD^2 = CP^2 - a^2 + b^2$$

$$= x^2 + \frac{b^2}{a^2}x^2 - b^2 - a^2 + b^2$$

$$= e^2x^2 - a^2 = (ex + a) \cdot (ex - a)$$

$$= SP \cdot HP.$$

200. To find the equation to the hyperbola referred to any system of conjugate diameters as axes.

The equation to a hyperbola referred to its center and axes is

$$a^2y^2 - b^2x^2 = -a^2b^2.$$

Let the conjugate diameters CP, CD (fig. 60) be the new axes of x' and y' inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes will be (Art. 42)

$$x = x' \cos\alpha + y' \cos\beta, \quad y = x' \sin\alpha + y' \sin\beta.$$

Hence, substituting and reducing,

$$(a^2 \sin^2\alpha - b^2 \cos^2\alpha)x'^2 + (a^2 \sin^2\beta - b^2 \cos^2\beta)y'^2$$

$$+ 2x'y(a^2 \sin\alpha \times \sin\beta - b^2 \cos\alpha \times \cos\beta) = -a^2b^2;$$

but $\tan\alpha \cdot \tan\beta = \frac{b^2}{a^2}$; therefore $a^2 \sin\alpha \cdot \sin\beta - b^2 \cos\alpha \cdot \cos\beta = 0$.

Also if $CP = a'$, $CD = b'$, we have (Art. 173)

$$a'^2(a^2 \sin^2\alpha - b^2 \cos^2\alpha) = -a^2b^2,$$

$$b'^2(a^2 \sin^2\beta - b^2 \cos^2\beta) = +a^2b^2;$$

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hence substituting and dividing by $-a'^2b'^2$, we get for the required equation

$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1,$$

or, in a geometrical form,

$$QV^2 = \frac{CD^2}{CP^2} PV . VG.$$

201. This equation, which, suppressing the accents of the variables, is

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2,$$

being of precisely the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the axes of the hyperbola and its conjugate diameters.

Hence the equation to the tangent at a point $Q(x', y')$ will be

$$a'^2yy' - b'^2xx' = -a'^2b'^2,$$

and if the tangent meet the axis of x in T, we shall have $CT = \frac{CP^2}{CV}$, as before; and if we wish to draw a tangent through an external point $Q(h, k)$ (fig. 61) we shall have, to determine the points of contact (x', y'), the equations

$$a'^2y'^2 - b'^2x'^2 = -a'^2b'^2,$$

$$a'^2ky' - b'^2hx' = -a'^2b'^2,$$

and if we construct the line represented by the latter, considering x' and y' as the variables, by taking $CT = \frac{a'^2}{h}$, $CR = \frac{-b'^2}{k}$ and joining RT, it will cut the hyperbola in the two points of contact.

202. Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any

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other point in this line we draw a pair of tangents to the hyperbola, the secant passing through the new points of contact will cut the diameter CP in T, as this point only changes when h changes. Hence from the several points of a straight line if pairs of tangents be drawn to a hyperbola, the straight lines which join the corresponding points of contact will all intersect in the same point.

203. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$\pm a'^2yy' - b'^2xx' = -a'^2b'^2,$$

according as we consider the point Q (x', y'), or the other extremity of the chord Q' whose co-ordinates are x', $-y'$; and in both cases when $y = 0$, $x = \frac{a'^2}{x'}$, therefore the tangents meet the axis of x in the same point T. (fig. 60).

204. If from any point within or without a hyperbola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangle of the segments will be to each other in an invariable ratio.

Let O (fig. 62) be the given point with co-ordinates h and k, then taking O for the pole, and measuring θ from a line parallel to the transverse axis, the equation to the hyperbola will be

$$a^2(r\sin\theta + k)^2 - b^2(r\cos\theta + h)^2 = -a^2b^2,$$

which is of the form $r^2 + Mr - N = 0$,

$$\text{where } N = \frac{-a^2b^2 - a^2k^2 + b^2h^2}{a^2\sin^2\theta - b^2\cos^2\theta} = r'r'',$$

if these be the two values of r.

Now let Pp, Qq, be drawn parallel to CP', CQ', which make angles α, β, with Cx, then

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$$PO \times Op : QO \times Oq :: a^2 \sin^2\beta - b^2 \cos^2\beta : a^2 \sin^2\alpha - b^2 \cos^2\alpha$$
$$:: CP'^2 : CQ'^2,$$

which ratio is independent of the position of the point O.

The Hyperbola referred to its Asymptotes.

205. The asymptotes of the hyperbola coincide with the diagonals of the parallelogram constructed upon any pair of conjugate diameters.

The equation to the diagonal CL (fig. 63) referred to the conjugate diameters CP, CD, is

$$y = \frac{b'}{a'} x.$$

And the equation to the hyperbola,

$$y = \frac{b'}{a'} \sqrt{x^2 - a'^2};$$

$$\therefore RQ = \frac{b'}{a'} (x - \sqrt{x^2 - a'^2}) = \frac{a'b'}{x + \sqrt{x^2 - a'^2}}.$$

Hence when $x = \infty$, RQ becomes zero; hence Ll' is an asymptote to the portions PQ, $P'Q'$; and similarly it may be shewn that $L'l$ is an asymptote to the other portions. Hence the asymptotes may be considered as the limits of the tangents (Art. 198).

206. If any chord of a hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.

Let Qq (fig. 63) any chord, when produced cut the asymptotes in R, r; bisect Qq in V, join CV, and refer the hyperbola to the diameter CP and its conjugate CD, then the equations to CR, Cr are

$$y = \frac{b'}{a'} x, \quad y = -\frac{b'}{a'} x;$$

$$\therefore VR = Vr, \text{ and } VQ = Vq; \therefore QR = qr.$$

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207. If a line be drawn through P parallel to Rr, it will be a tangent at P, and $PL = Pl$.

$$\text{Also } RQ \times Qr = RV^2 - QV^2 = \frac{b'^2}{a'^2}\{x^2 - (x^2 - a'^2)\} = b'^2 = CD^2.$$

Hence when a line cutting the hyperbola is parallel to a diameter, the rectangle of the parts of this secant between a point of the curve and the asymptotes is equal to the square of the semi-diameter.

208. From any point P (fig. 63) of the hyperbola, draw parallels PG, PF to the asymptotes, and draw the tangent Ll which is bisected in P. Then the parallelogram GF is half of the triangle LCl.

But area LCl is constant, whatever be the position of P, and equals ab (Art. 196).

Hence, denoting by 2α the angle between the asymptotes, and by x, y, the co-ordinates of P referred to the asymptotes as axes, so that $CF = x$, $FP = y$, we get

$$xy \sin 2\alpha = \frac{ab}{2}; \quad \text{but } \sin 2\alpha = \frac{2\tan\alpha}{1 + \tan^2\alpha} = \frac{2ab}{a^2 + b^2};$$

$$\therefore\ xy = \tfrac{1}{4}(a^2 + b^2),$$

the equation to the hyperbola referred to its asymptotes.

209. To transform the equation to the hyperbola referred to its axes, into that representing the hyperbola referred to its asymptotes.

Let the inferior asymptote be the axis of x'; then since the axes of x' and y' make with the axis of x the angles $2\pi - \alpha$ and α respectively, the formulæ are (Art. 42)

$$x = x' \cos\alpha + y' \cos\alpha, \quad y = y' \sin\alpha - x' \sin\alpha;$$

$$\therefore\ a^2 \sin^2\alpha\,(y' - x')^2 - b^2 \cos^2\alpha\,(y' + x')^2 = -a^2b^2,$$

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but $\tan \alpha = \frac{b}{a}$; and $\therefore a^2 \sin^2 \alpha = b^2 \cos^2 \alpha = \frac{b^2}{1 + \frac{b^2}{a^2}} = \frac{a^2 b^2}{a^2 + b^2}$;

$$\therefore \frac{a^2 b^2}{a^2 + b^2} \cdot 4x'y' = a^2 b^2, \text{ or } x'y' = \frac{a^2 + b^2}{4}.$$

210. To find the equation to the line touching the hyperbola at a given point, when referred to its asymptotes.

Let the co-ordinates of the given point be x', y', and those of a point near it x'', y''; the equation to the line passing through them will be

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

but $x'y' = \frac{1}{4}(a^2 + b^2)$, $x''y'' = \frac{1}{4}(a^2 + b^2)$;

$$\therefore x''y'' - x'y' = 0, \text{ or } x'(y'' - y') + y''(x'' - x') = 0;$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{y''}{x'}.$$

Hence the equation to the secant becomes

$$y - y' = -\frac{y''}{x'}(x - x'),$$

and in order that it may become a tangent we must suppose $y'' = y'$ which gives

$$y - y' = -\frac{y'}{x'}(x - x').$$

211. To find where it cuts the asymptotes, make $y = 0$; $\therefore x = 2x'$,

or $Cl = 2CF$, (fig. 63).

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SECTION IX.

ON THE SECTIONS OF THE CONE AND CYLINDER.

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212. THE surface described by an indefinite straight line which is carried round the perimeter of a given circle, always passing through a fixed point, is called a cone (fig. 64).

The circle is called the base of the cone, and the fixed point its vertex, and the line joining the vertex and center of the base is called the axis. The cone is moreover right or oblique according as the axis is at right angles, or inclined, to the plane of the base.

As the generating line is unlimited in both directions from the vertex, the surface of the cone is composed of two portions or sheets, perfectly similar, situated on opposite sides of the vertex. Also from the mode of generation it follows that every plane parallel to the base will cut the cone in a circle; and every plane through the axis will cut it in two straight lines. When the surface is a right cone, every generating line will make the same angle with the axis. The different curves obtained by cutting a cone by a plane are called Conic Sections.

213. All conic sections are curves of the second order.

Let PAP' (fig. 65) be a section of a right cone made by any plane, and through the axis VO draw a plane perpendicular to that of the section, cutting the cone in the lines VB, VD, and the plane of the section in the line AN, which take for the axis of x. Through any point P draw a plane perpendicular to the axis, intersecting the cone in the circle MPQ, and the plane of the section in PP';

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then MQ will be a diameter of the circle, and PN will be at right angles to both AN and NM, and will consequently be a common ordinate to the circle and conic section AP. Draw Ay parallel to PN and take it for the axis of y, and let $AN = x$, $PN = y$; and choosing the data so as to embrace every case, let $AV = d$, $\angle VAN = \theta$, $AVQ = 2a$. Then

$$y^2 = MN \times NQ.$$

$$\text{But } \frac{MN}{AN} = \frac{\sin\theta}{\cos a}; \qquad \therefore\ MN = \frac{x \sin\theta}{\cos a}.$$

And drawing NF parallel to QV,

$$\frac{AF}{AN} = \frac{\sin(2a+\theta)}{\cos a}; \qquad \therefore\ AF = \frac{x \sin(2a+\theta)}{\cos a};$$

$$\therefore\ NQ = 2d \,.\, \sin a - \frac{x \,.\, \sin(2a+\theta)}{\cos a};$$

$$\therefore\ y^2 = \frac{2d \,.\, \sin a \,.\, \sin\theta}{\cos a} x - \frac{\sin\theta \,.\, \sin(2a+\theta)}{\cos^2 a} x^2,$$

the equation to a curve of the second order; therefore every conic section is a curve of the second order; and it will be an ellipse, hyperbola, or parabola, according as the second term is negative, or positive, or zero.

Now the second term can only change its sign when $\sin(2a+\theta)$ changes its sign. Hence the section will be an ellipse as long as $2a+\theta$ is less than π, and therefore AN meets VQ produced, or the cutting plane meets only one sheet of the cone.

It will be a hyperbola when $2a+\theta$ is greater than π, and therefore AN and VQ intersect when produced backwards, and the cutting plane meets both the sheets of the cone.

It will be a parabola when $2a+\theta = \pi$, and therefore AN, VQ are parallel, or the cutting plane is parallel to a generating line of the cone.

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214. To determine the axes of the conic section, we have, comparing the equation (supposing it to represent an ellipse, and therefore $\sin(2\alpha+\theta)$ to be positive)

$$y^2 = \frac{2d\sin\alpha\,.\,\sin\theta}{\cos\alpha}\,x - \frac{\sin\theta\,.\,\sin(2\alpha+\theta)}{\cos^2\alpha}\,x^2$$

$$\text{with } y^2 = \frac{2b^2}{a}\,x - \frac{b^2}{a^2}\,x^2,$$

the latus rectum or $\dfrac{2b^2}{a} = 2d\,.\,\tan\alpha\,.\,\sin\theta,$

$$\text{and } \frac{b^2}{a^2} = \frac{\sin\theta\,.\,\sin(2\alpha+\theta)}{\cos^2\alpha};$$

$$\therefore\ 2a = \frac{2d\sin\alpha\,.\,\cos\alpha}{\sin(2\alpha+\theta)};$$

$$\therefore\ 2b^2 = \frac{2d^2\sin^2\alpha\sin\theta}{\sin(2\alpha+\theta)} \text{ or } b = d\,.\,\sin\alpha\sqrt{\frac{\sin\theta}{\sin(2\alpha+\theta)}}.$$

215. The minor axis may however be more conveniently expressed in the following manner.

From the extremities of the axis major let fall perpendiculars $AF=f$, $A'G=g$ (fig. 64), upon the axis of the cone; through C the middle point of AA' draw a plane parallel to the base, cutting the section in BB' which is its minor axis, and the cone in the circle MBQ, then

$$BC^2 = MC \times CQ = A'G \times AF = fg,$$

because MC, being parallel to DA', $= \frac{1}{2}DA' = A'G$,

and similarly $CQ = AF$.

Hence the distance of the foci of the elliptic section $= AD$;

for dropping the perpendicular AE, $A'E = f+g$;

$$\therefore\ AD^2 = 4a^2 + 4g^2 - 4g(f+g) = 4a^2 - 4fg = 4(a^2-b^2);$$

$$\therefore\ AD = 2\sqrt{a^2-b^2} = \text{distance of foci.}$$

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216. If in that section of a cone through the axis which is perpendicular to the plane of an elliptic section, we describe circles touching the generating lines of the cone and the axis of the section, the points of contact with the axis will be the foci of the section.

For the distance of the foci $= A'D'$ (fig. 66).

But $A'D' = A'U' - D'U' = A'S - AU$

$= AA' - 2AS$

$\therefore \quad AS = \frac{1}{2}(AA' - A'D')$;

therefore S is a focus. Similarly, H may be shewn to be the other focus.

Produce UU' to meet AA' produced in X,

then from the similar triangles AUX, ADA',

$$\frac{AX}{AA'} = \frac{AU}{AD} \quad \text{or} \quad \frac{AX}{2AC} = \frac{AS}{2SC};$$

$$\therefore \quad \frac{AX}{AC} = \frac{AS}{SC} \quad \text{or} \quad \frac{CX}{AC} = \frac{AC}{SC};$$

therefore X is the point where the directrix meets the axis (Art. 110). Similarly, X' is the point where the other directrix meets the axis.

217. When the section is a hyperbola the equation is

$$y^2 = \frac{2d \sin\theta \,.\, \sin\alpha}{\cos\alpha} x - \frac{\sin\theta \,.\, \sin(2\alpha + \theta)}{\cos^2\alpha} x^2,$$

where $\sin(2\alpha + \theta)$ is a negative quantity, and consequently the second term is positive, which compared with

$$y^2 = \frac{2b^2}{a} x + \frac{b^2}{a^2} x^2$$

will determine the axes, as in the case of the ellipse. When $d = 0$ the equation becomes

$$y^2 = -\frac{\sin\theta \,.\, \sin(2\alpha + \theta)}{\cos^2\alpha} x^2,$$

which represents two generating lines of the cone.

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In this case also the semi-conjugate axis is a mean proportional between the perpendiculars dropped from the vertices of the hyperbola upon the axis of the cone.

218. When the section is a parabola, or $2\alpha + \theta = \pi$, the equation is

$$y^2 = \frac{2d \sin\theta \cdot \sin\alpha}{\cos\alpha} x = 4d \cdot \sin^2\alpha \cdot x, \text{ since } \sin\theta = \sin 2\alpha.$$

219. We must now demonstrate the converse proposition, namely, that curves of the second order are conic sections.

Every curve of the second order is contained in the equation

$$y^2 = 4px + nx^2,$$

where $4p$ is the latus rectum, and n the square of the ratio of the axes, abstracting the sign. What we have to demonstrate is, that the quantities p, n, and α being given, we can assign real values of d and θ which shall render the above equation identical with

$$y^2 = \frac{2d \sin\alpha \cdot \sin\theta}{\cos\alpha} x - \frac{\sin\theta \cdot \sin(2\alpha + \theta)}{\cos^2\alpha} x^2.$$

Equating the coefficients of x and x^2 in the two equations, we get

$$\frac{d \sin\alpha \cdot \sin\theta}{\cos\alpha} = 2p, \quad \frac{\sin\theta \cdot \sin(2\alpha + \theta)}{\cos^2\alpha} = -n,$$

the former of which will give a real value of d when θ is real; the latter may be transformed into

$$\tfrac{1}{2}\{\cos 2\alpha - \cos(2\alpha + 2\theta)\} = -n\cos^2\alpha,$$

$$\text{or } \cos 2(\alpha + \theta) = 2(1 + n)\cos^2\alpha - 1.$$

In the ellipse n is negative and less than 1; hence the preceding value of $\cos 2(\alpha + \theta)$ lies between $+1$ and -1, and therefore θ is always real. Consequently any given ellipse

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may be regarded as a section of any proposed right cone whatever.

In the hyperbola n is positive and of any magnitude; if the above value of $\cos 2(\alpha+\theta)$ be negative, it will be evidently less than 1, and α will be real; but if it be positive, we must have

$$2(1+n)\cos^2\alpha - 1 \text{ less than } 1,$$

$$\text{and } \therefore \cos\alpha \text{ less than } \frac{1}{\sqrt{1+n}}, \text{ or than } \frac{a}{\sqrt{a^2+b^2}};$$

but if ω be the angle which the asymptote makes with the transverse axis, $\cos\omega = \frac{a}{\sqrt{a^2+b^2}}$; $\therefore \cos\alpha < \cos\omega$, $\therefore \alpha > \omega$; and therefore in order that a given hyperbola may be cut from a given cone, the vertical angle of the cone must be not less than the angle between the asymptotes.

In the parabola $n=0$, therefore $\sin\theta = 0$ or $\sin(2\alpha+\theta)=0$; the first is inadmissible, for it makes $p=0$; the second gives $2\alpha+\theta=\pi$, which will always furnish a real value for θ; hence a given parabola may be cut from any proposed cone.

220. To determine the curve which results from the intersection of a right cylinder by a plane.

Let APA' (fig. 67) be a section of a right cylinder, $AA'D$ a section of the cylinder through its axis, perpendicular to the plane of the section. Through any point P draw a plane perpendicular to the axis of the cylinder, intersecting it in a circle whose diameter is MQ, and the plane of the section in PP' which will be perpendicular to MQ, AA', and will be a common ordinate of the section and circle.

Let $AN = x$, $NP = y$, $AA' = 2a$, $AD = 2r$,

then $y^2 = MN \cdot NQ$;

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but $\frac{MN}{AD} = \frac{AN}{AA'}$, or $MN = \frac{r}{a}x$,

$$\frac{NQ}{AD} = \frac{NA'}{AA'}, \quad \text{or } NQ = \frac{r}{a}(2a - x);$$

$$\therefore y^2 = \frac{r^2}{a^2}(2ax - x^2),$$

the equation to an ellipse.

221. In the same manner the nature of the sections of an oblique cone may be determined; but this, as well as the discussion of the sections of Conoids or figures generated by the revolution of conic sections about their axes, may be more conveniently deferred to Geometry of Three Dimensions. There is however one property of the oblique cone which admits of a simple demonstration, viz. that it may be cut by other planes besides those parallel to its base, so that the sections may be circles, and which we shall give here.

Let VBD (fig. 68) be the principal section of an oblique cone, that is, a section made by a plane through its axis perpendicular to its base; and let MPQ, APA' be two sections made by planes perpendicular to BVD, and of which the former is parallel to the base, and is therefore a circle with diameter MQ, and as PN is perpendicular to MQ, we have $PN^2 = MN \cdot NQ$; and the latter will also be a circle if angle $AA'V = ABD$, for in that case the triangles AMN, $A'NQ$ are similar, and $\frac{NA'}{NQ} = \frac{MN}{NA}$;

$$\therefore A'N \cdot NA = MN \cdot QN = PN^2,$$

and as PN is perpendicular to AA', the section, which is called a subcontrary section, is a circle; its plane is perpendicular to the principal section of the cone, and makes the same angle with one of the generating lines of the cone which are in that plane, as the plane of the base does with the other.

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SECTION X.

ON THE GENERAL EQUATION OF CURVES OF THE SECOND ORDER, AND ON CERTAIN GENERAL PROPERTIES OF ALGEBRAICAL CURVES.

---

222. We shall now proceed to the reduction of the general equation of the second degree

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

where we suppose the co-ordinates rectangular; for if they were oblique, by transforming them to rectangular co-ordinates we should obtain an equation of the same degree as the above, and which could not therefore be more general than the one we have assumed. We shall prove, as affirmed at Art. 62, that this equation by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms,

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

the co-ordinates being rectangular; and therefore can never represent any other curve than one of those discussed in the preceding Sections. The principle of the method is to change the system of co-ordinates, without giving any particular values to the quantities which determine the position of the new axes. By that means, indeterminate quantities are introduced into the transformed equation, to which such values can afterwards be assigned as will destroy certain of its terms. Instead of altering both the origin and direction of the co-ordinate axes at once, it is more convenient to effect these changes separately, in the following manner.

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225. We shall now proceed to the actual determination of the axes of the curve. Since

$$\sec 2\theta = 1 + \frac{b^2}{(a-c)^2}, \qquad \tan 2\theta = \frac{-b}{a-c};$$

$$= \frac{(a-c)^2 + b^2}{(a-c)^2}; \quad \therefore \cos = \frac{1}{\sec} = \frac{1}{\sqrt{\quad}}$$

$$\therefore \cos 2\theta = \frac{a-c}{\sqrt{(a-c)^2 + b^2}},$$

$$\sin 2\theta = \cos 2\theta \,.\, \tan 2\theta = \frac{-b}{\sqrt{(a-c)^2 + b^2}},$$

in these expressions the radical may have either the sign + or −, because we are at liberty to choose either of the new axes for the axis of x; but to avoid all ambiguity, we shall take the radical with a positive sign; then $\sin 2\theta$ will have a sign contrary to that of b.

Hence taking the sum and difference of equations (2), and substituting the above values of $\cos 2\theta$ and $\sin 2\theta$, we get

$$A + B = a + c,$$

$$A - B = (a-c)\cos 2\theta - b \sin 2\theta = \frac{(a-c)^2 + b^2}{\sqrt{(a-c)^2 + b^2}} = \sqrt{(a-c)^2 + b^2};$$

$$\therefore A = \tfrac{1}{2}\{a + c + \sqrt{(a+c)^2 + m}\},$$

$$B = \tfrac{1}{2}\{a + c - \sqrt{(a+c)^2 + m}\},$$

putting $m = b^2 - 4ac$.

226. We have now two cases to consider, according as m is positive or negative.

First let m be negative, then A and B have the same sign; and supposing $\phi(h, k)$ to be of a contrary sign to A and B, and $= -C$, the equation is

$$Ay^2 + Bx^2 = C,$$

which represents an ellipse with semi-axes $\sqrt{\frac{C}{A}}$, $\sqrt{\frac{C}{B}}$.

If $\phi(h, k) = 0$, the equation is satisfied only by $x = 0$, $y = 0$, i. e. it represents the point which is the origin; and if $\phi(h, k)$ be of the same sign as A and B, the equation can be satisfied by no real values of x and y.

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Secondly, let m be positive, then A and B have contrary signs; and whatever be the sign of $\phi(h, k)$ i.e. whether it equals $+C$ or $-C$, the equation will be of one of the forms

$$\frac{Ay^2}{C} - \frac{Bx^2}{C} = 1, \quad \text{or} \quad \frac{Bx^2}{C} - \frac{Ay^2}{C} = 1,$$

which represents a hyperbola with semi-axes $\sqrt{\frac{C}{A}}$, $\sqrt{\frac{C}{B}}$.

If $C = 0$, the equation is $y = \pm\sqrt{\frac{B}{A}} \cdot x$, which represents two straight lines through the origin.

227. Next in the equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

let the coefficients be such that $b^2 - 4ac = 0$, and that the numerators in the values of h and k are finite, then the coordinates of the center are infinite, which signifies that the curve has no center. In this case, as we cannot by changing the origin take away the terms in x and y, our first object must be to destroy the term involving the rectangle xy. For that purpose put

$$x' \cos\theta - y' \sin\theta \text{ for } x, \text{ and } x' \sin\theta + y' \cos\theta \text{ for } y,$$

and the equation becomes

$$Ay'^2 + Bx'^2 + (d\cos\theta - e\sin\theta)\, y' + (d\sin\theta + e\cos\theta)\, x' + f = 0,$$

the term involving $x'y'$ disappearing, as before, by the condition

$$\tan 2\theta = \frac{-b}{a-c},$$

which gives, since $b^2 = 4ac$,

$$\sin 2\theta = \frac{-b}{\sqrt{(a-c)^2 + b^2}} = \frac{-b}{a+c}$$

$$\cos 2\theta = \frac{a-c}{a+c},$$

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taking the radical with the positive sign. Hence by means of the formulæ

$$\cos\theta = \sqrt{\tfrac{1}{2}(1+\cos 2\theta)},\quad \sin\theta = \sqrt{\tfrac{1}{2}(1-\cos 2\theta)},\text{ we get}$$

$$d\cos\theta - e\sin\theta = \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a+c}} = D,$$

$$d\sin\theta + e\cos\theta = \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a+c}} = E.$$

$$\text{Also } A = \tfrac{1}{2}\left\{a+c+\sqrt{(a-c)^2+b^2}\right\} = a+c,$$

$$B = \tfrac{1}{2}\left\{a+c-\sqrt{(a-c)^2+b^2}\right\} = 0,$$

therefore the equation becomes, suppressing the accents,

$$Ay^2 + Dy + Ex + f = 0,$$

$$\text{or } \left(y + \tfrac{1}{2}\frac{D}{A}\right)^2 = \frac{E}{A}\left(-x + \frac{D^2 - 4Af}{4AE}\right),$$

which represents a parabola, latus rectum $= \frac{E}{A}$, and co-ordinates of its vertex $x = \frac{D^2 - 4Af}{4AE}$, $y = -\frac{1}{2}\frac{D}{A}$, and axis parallel to the new axis of x. In this case the co-ordinates of the new origin cannot become infinite; for $A = a+c$ cannot become zero since a and c have the same sign; and if $E = 0$, then the transformed equation will no longer contain x; and being solved with respect to y, it will furnish two constant values for y, so that it will represent two parallel lines.

228. If in Art. 223 the coefficients of the proposed equation are such that one of the numerators $2ae - bd$ is zero, at the same time that $b^2 = 4ac$, (which two suppositions make the other numerator $2cd - be$ also vanish) both the co-ordinates of the center become indeterminate; the two equations (1) in that case are equivalent to a single independent equation, and the two lines which they represent, regarding h and k as the co-ordinates, coincide, and there

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exists an infinite number of centers all situated in that line. The proposed equation, with the above relations among its coefficients, no longer in fact represents a curve, but two parallel straight lines; for, solving it, we get

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{(b^2-4ac)x^2+2(bd-2ac)x+d^2-4af},$$

and this in the supposed case becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{d^2-4af},$$

and therefore represents two parallel straight lines, which are replaced by a single one if $d^2 = 4af$; and become altogether imaginary if $d^2 < 4af$.

General properties of Algebraical Curves.

229. The general equation of the n^{th} degree between x and y ought to contain all the combinations of the powers of x and y in which the sum of the indices does not exceed n; therefore when complete and arranged according to descending powers of y, it will be

$$a_0y^n + (b_0 + b_1x)\,y^{n-1} + (c_0 + c_1x + c_2x^2)\,y^{n-2} + \&c.$$
$$+ (l_0 + l_1x + l_2x^2 + \&c. + l_nx^n) = 0.$$

All equations between two variables x and y which can be reduced to this form, are called algebraical, all others are called transcendental; hence arises the distinction of lines into algebraical and transcendental, according as their equations are algebraical or transcendental.

230. The classification of lines in different orders according to the degrees of their equations would be to little purpose, if by changing the axes of the co-ordinates we altered the degree of the equation. But this is not the case. For having given, between x and y, the equation to a line referred to certain axes, in order to get the equation to the same line referred to new axes, we must replace x and y in the given equation by the values found in Art. 41; and as these values are of the first degree in x' and y', it

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follows that the degree of the equation cannot be raised by this substitution. Neither can the degree of the transformed equation be less than that of the primitive equation; for if it could, then, by what has been proved, we could not return from it to the primitive equation, which is absurd.

231. The general equation of any degree comprehends not only all lines of the order expressed by that degree, but also all lines of inferior orders. Thus the above general equation of the n^{th} degree, by making $a_0 = b_1 = c_2 = \ldots = l_n = 0$, degenerates into the equation of the $(n-1)^{\text{th}}$. Also the equation of the second degree

$$(y - mx - c)(y - m'x - c') = 0$$

is clearly verified either by putting $y = mx + c$, or $y = m'x + c'$, which represent two lines of the first order; so that the proposed equation does not in reality represent a line of the second order at all, but two straight lines; or only one even of these, if $m = m'$, $c = c'$. Similarly, the equation of the third order

$$(y - mx - c)(ay - x^2) = 0$$

represents a line of the first order, and one of the second whose equation is $ay - x^2 = 0$. And in general according as a proposed equation of any degree is not, or is capable of being resolved into factors which are rational with respect to the variables x and y, it will represent a single line of the corresponding order, or several distinct lines of inferior orders.

232. A straight line cannot meet a curve of the n^{th} order in more than n points.

Let the co-ordinates be transformed so that the proposed line may be the axis of x, and let $V = 0$ be the resulting equation to the curve; in order to determine the points in which it is intersected by the straight line, we must put $y = 0$ in the equation $V = 0$, and the corresponding values of x will be the abscissæ of the required points. But $V = 0$ being of the n^{th} degree, the equation for determining x will be at the most of the n^{th} degree; therefore

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x cannot have more than n values, and there cannot be more than n points of intersection; but there may be fewer than n, for the equation for determining x may be of a degree inferior to n, and may have equal or imaginary roots.

233. The general equation of the n^{th} degree between two variables, when complete, contains $1 + 2 + 3 + \&c. + (n + 1)$, or $\frac{1}{2}(n + 1)(n + 2)$, arbitrary constants, in which, since we may divide the whole equation by one of them, there is one superfluous which might be suppressed; consequently the number of independent constants is

$$\frac{1}{2}(n + 1)(n + 2) - 1, \text{ or } \frac{1}{2}n(n + 3).$$

Hence a curve of the n^{th} order may be made to fulfil $\frac{1}{2}n(n + 3)$ conditions; as, for instance, to pass through $\frac{1}{2}n(n + 3)$ points; for giving to x and y their values at each of the given points, we get $\frac{1}{2}n(n + 3)$ different equations by means of which the values of the constants may be determined. Hence a curve of the second order may pass through five given points.

234. To determine the conic section which passes through five given points.

Take the axes of the co-ordinates so that each contains two of the given points; and let y_1, y_2, be the ordinates of the points situated in the axis of y; x_1, x_2, the abscissæ of the points situated in the axis of x; and x_3, y_3, the co-ordinates of the fifth given point. Then substituting successively the co-ordinates of each of these points in the place of x and y in the general equation (where every coefficient is divided by the constant term),

$$ay^2 + bxy + cx^2 + dy + ex + 1 = 0,$$

we get the five equations

$$ay_1^2 + dy_1 + 1 = 0, \qquad ay_2^2 + dy_2 + 1 = 0,$$
$$cx_1^2 + ex_1 + 1 = 0, \qquad cx_2^2 + ex_2 + 1 = 0,$$
$$ay_3^2 + bx_3y_3 + cx_3^2 + dy_3 + ex_3 + 1 = 0,$$

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which give for the five unknown quantities the values

$$a = \frac{1}{y_1 y_2}, \quad d = -\frac{y_1 + y_2}{y_1 y_2}, \quad c = \frac{1}{x_1 x_2}, \quad e = -\frac{x_1 + x_2}{x_1 x_2},$$

$$b = -\frac{1}{x_3 y_3}\left\{\frac{y_3 (y_3 - y_2 - y_1)}{y_1 y_2} + \frac{x_3 (x_3 - x_2 - x_1)}{x_1 x_2} + 1\right\}.$$

Now provided no three of the given points be in a straight line, none of the quantities x_1, x_2, &c. is zero; therefore the above values of a, b, &c. are neither infinite, nor indeterminate, and none of them has more than one value; therefore through five points, provided no three be in a straight line, a conic section, and only one, can be made to pass.

235. To find the position of the center of any curve.

The center of a curve is a point C (fig. 24), such that any chord of the curve PP' drawn through it, is bisected in it. (It must be observed, however, that if PP' meet the curve in more points than two, it is sufficient that these points combined in a certain order should be two and two equally distant from C). If the curve be referred to any two axes originating in C, and PN, $P'N'$ be the ordinates parallel to Cy of the extremities of a chord, we see from the equal triangles PCN, $P'CN'$, that these ordinates are equal and of contrary signs; the same thing is true for the abscissæ of P and P'; as well as for the extremities of every other chord passing through C. If therefore $\phi(x, y) = 0$ be the equation to the curve, and if it be satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = -b$; that is, it must be such as not to alter when the signs of the two variables are changed; and conversely, if it have this property, the origin is the center of the curve. When $\phi(x, y) = 0$ is algebraic, it cannot have the above property unless the dimension of every term be even in an equation of an even degree, and odd in an equation of an odd degree; for in the former case the equation is not at all altered by replacing x and y by $-x$ and $-y$; and in the latter (in which case the equation cannot have a constant term) the sign of every

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term will be altered, and therefore the whole equation unaltered. Hence to find whether a proposed curve admits of a center, we must refer it to parallel axes through a new origin having co-ordinates h, k, by putting $x = x' + h$, $y = y' + k$, and equate to zero the coefficients of all the terms which are of a dimension different (as far as regards odd and even) from the degree of the equation; if these conditions can all be satisfied by real and finite values of h and k, the curve has a center, and h and k are its co-ordinates; in the contrary case the curve has no center. Of this process we have an example at Art. 223.

236. The locus of the middle points of a system of parallel chords of any curve is called its diametral curve. If the curve be of the n^{th} order, the points of intersection with its ordinates real or imaginary will be in number n; and their combinations on the same indefinite line will form $\frac{1}{2}n(n-1)$ different chords, and as many middle points, and therefore the diametral curve, since it may be met by an indefinite line in $\frac{1}{2}n(n-1)$ points, will have an equation of the degree $\frac{1}{2}n(n-1)$. For curves of the second order, since $n = 2$, the diametral curves can only be straight lines.

237. To find the locus of the middle points of a system of parallel chords of any curve.

Let the chords be parallel to a line through the origin whose equation is $y = mx$, and let $\phi(x, y) = 0$, be the equation to the curve; also let x', y', be the co-ordinates of the middle point of any one of these chords, and take it for the origin without altering the direction of the axes, and therefore put $x' + x$ for x, and $y' + y$ for y; then the transformed equation to the curve is $\phi(x' + x, y' + y) = 0$, and the equation to the chord is $y = mx$. Hence the values of x, corresponding to the points of intersection of the curve and chord, result from the equation $\phi(x' + x, y' + mx) = 0$, and because the origin bisects the chord, the values of x must be equal and of contrary signs; from which consideration we may obtain a relation between x' and y', which is the equation to the required locus.

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238. Thus if $\phi(x, y) = 0$ be the general equation of the second order,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

putting $x = x' + x$, and $y = y' + mx$, we get

$$a(y' + mx)^2 + b(x + x')(y' + mx) + c(x + x')^2$$
$$+ d(y' + mx) + e(x + x') + f = 0,$$

and because the values of x are to be equal and of contrary signs, the term involving the first power of x must disappear;

$$\therefore\ 2amy' + b(x'm + y') + 2cx' + dm + e = 0,$$
$$\text{or } y'(2am + b) + x'(2c + bm) + dm + e = 0,$$

the equation to a straight line. Hence there will be an infinite number of diameters corresponding to the various values of m. If the diameter is to be perpendicular to its chords, we must have

$$1 + m\left(-\frac{2c + bm}{2am + b}\right) = 0, \text{ or } m^2 + \frac{2(c - a)}{b}m - 1 = 0,$$

which will necessarily give two real values of m; hence there are generally two diameters which bisect their ordinates perpendicularly.

239. Not only do we distribute algebraical curves, into orders according to the degree of their equations; but we also investigate the different families of lines which may be comprised amongst those of the same order; and even the different species of each family, if necessary. The individual lines of the same family, or species, we then classify according to certain characteristics easy to be recognized, which completely distinguish them from one another; and we lastly endeavour to determine the form and properties of each of them. This we have effected for equations of the first and second degrees; the former gives only straight lines as has been said; the latter gives three species of curves very distinct, viz. the parabola which has no center; and the ellipse and hyperbola, both of which have a center, but only the latter has asymptotes.

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The enumeration of lines of the third order was first made by Newton, who found 72 species comprised in 14 divisions; Stirling added 4 species which had been omitted, and, lastly, Cramer added 2 more, making in all 78 species.

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240. The following are examples of finding the equations to various loci, and of tracing curves by means of their equations.

(1) Having given the base and altitude of a triangle, to find the locus of the center of the inscribed circle.

$SC = CH = c$ = half given base (fig. 41), $PN = a$ the given altitude, O the center of the inscribed circle, $CM = x$, $MO = y$ its co-ordinates, $CN = x'$; then

$$\tan S = \frac{a}{c+x'}, \quad \tan\frac{S}{2} = \frac{y}{c+x}, \quad \tan H = \frac{a}{c-x'}, \quad \tan\frac{H}{2} = \frac{y}{c-x};$$

$$\therefore \frac{a}{c+x'} = \frac{2y(c+x)}{(c+x)^2-y^2}, \quad \frac{a}{c-x'} = \frac{2y(c-x)}{(c-x)^2-y^2};$$

therefore inverting and adding in order to eliminate x', we get the required equation, which is of the third degree,

$$\frac{2}{a} = \frac{1}{y} - \frac{y}{c^2-x^2}.$$

(2) Having given the base of a triangle, and the sum of the other two sides, to find the locus of the center of the inscribed circle.

SH the given base $= 2ae$, $SP + PH = 2a$, and $SO = r$, $HSO = \theta$, (fig. 41), polar co-ordinates of the describing point O;

then area of triangle = $\frac{1}{2}$ perimeter × radius of inscribed circle

$$= a(1+e)\, r \sin\theta,$$

also area of triangle $= SP \sin 2\theta \,.\, ae = \dfrac{a^2 e(1-e^2)\sin 2\theta}{1-e\cos 2\theta}$;

$$\therefore r = \frac{2ae(1-e)\cos\theta}{1-e\cos 2\theta},$$

the equation to a conic section of which SH is the major axis.

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(3) Two given circles are traced upon a plane, and a line is drawn touching one and cutting the other in two points, at which tangents are drawn to the latter circle; to find the locus of the intersection of the tangents.

$OP = r$, $POO' = \theta$, $OO' = c$, $OQ = a$, $O'Q' = a'$ (fig. 69),

$$c\cos\theta - a' = ON = \frac{OQ^2}{OP} = \frac{a^2}{r};$$

$$\therefore\ r = \frac{a^2}{c\cos\theta - a'},$$

hence the locus is a conic section of which the centers of the circles are the foci.

(4) Two focal distances of a conic section include a constant angle, and one of them is produced to meet the tangent at the extremity of the other, to find the locus of the point of intersection.

$AST = \theta$, $ST = r$, $PSQ = \alpha$, $SPT = \phi$ (fig. 30),

$$\frac{SP}{ST} = \frac{\sin(\alpha+\phi)}{\sin\phi} = \cos\alpha + \sin\alpha\cot\phi$$

$$= \cos\alpha + \sin\alpha\frac{e\sin(\theta+\alpha)}{1+e\cos(\theta+\alpha)} = \frac{\cos\alpha + e\cos\theta}{1+e\cos(\theta+\alpha)},$$

$$\because\ \cot\phi = \tan SPG = \frac{GL}{PL} = \frac{e\,SP\sin PSA}{a(1-e^2)} \text{ (Art. 135)},$$

$$\text{and } SP = \frac{a(1-e^2)}{1+e\cos(\theta+\alpha)};$$

$$\therefore\ r = \frac{a(1-e^2)}{\cos\alpha + e\cos\theta},$$

the equation to a conic section with focus S, and which is an ellipse, hyperbola, or parabola, according as $\cos\alpha >, <$ or $= e$.

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(5) To find the equation to the curve traced out by a point in the perimeter of a circle which rolls upon another equal circle.

Let A' be the describing point, at first in contact with A, and AA' the curve traced out, (fig. 70); C, C' the centers of the circles; join AA', CC', and let

$$AA' = r,\ A'AE = \theta,\ AC = a.$$

AA' is manifestly parallel to CC', draw DA' parallel to AC, and therefore $= AC$;

$$\therefore\ AA' = CD = CC' - DC',$$

or $r = 2a - 2a\cos\theta$, the polar equation.

Or if $AR = x$, $RA' = y$, be the rectangular co-ordinates of A',

$$\sqrt{x^2 + y^2} = 2a\left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right);$$

$$\therefore\ x^2 + y^2 = 2a\left(\sqrt{x^2 + y^2} - x\right).$$

(6) To find the equation to the curve described by a point in the perimeter of a circle which rolls within another circle of four times its radius.

P the describing point, at first in contact with A (fig. 70), and AP the curve traced out;

$$CN = x,\ PN = y,\ CA = 4a,\ QO = a,$$

O being the center of the rolling circle,

$ACD = \theta$; therefore $QOP = 4\theta$, and consequently, OM being perpendicular to AC,

$$\angle POM = \pi - \left(\frac{\pi}{2} - \theta\right) - 4\theta = \frac{\pi}{2} - 3\theta;$$

$$\therefore\ x = CM + Pn = 3a\cos\theta + a\cos 3\theta = 4a(\cos\theta)^3,$$

$$y = OM - On = 3a\sin\theta - a\sin 3\theta = 4a(\sin\theta)^3;$$

$$\therefore\ \left(\frac{x}{4a}\right)^{\frac{2}{3}} + \left(\frac{y}{4a}\right)^{\frac{2}{3}} = 1.$$

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(7) To find the locus of the intersection of a tangent to a given curve and the perpendicular let fall upon it from a given point.

PY (fig. 71) a tangent to the given curve at a point whose co-ordinates are $SN = x$, $PN = y$, S being the given point; SY a perpendicular upon PY, $SM = x'$, $MY = y'$, the co-ordinates of Y. Then drawing YQ parallel to SN, from the similar triangles SMY, PYQ, $\frac{SM}{MY} = \frac{PQ}{QY}$

$$\text{or } \frac{x'}{y'} = \frac{y - y'}{x + x'} \text{ or } x'^2 + y'^2 = yy' - xx'.$$

Also if α be the angle which PY forms with the axis of x, $\tan\alpha = f(x, y)$ is known from the nature of the curve, but $\tan\alpha = \frac{x'}{y'}$;

$$\therefore \frac{x'}{y'} = f(x, y).$$

These two equations together with the equation to the curve will enable us to eliminate x and y, and the result will be the equation to the locus of Y.

Ex. 1. To find the locus of the intersection of the tangent to a parabola and the perpendicular upon it from the vertex.

$$y^2 = 4ax,$$

$$\tan\alpha = \frac{2a}{y} = \sqrt{\frac{a}{x}}; \quad \therefore \frac{x'}{y'} = \sqrt{\frac{a}{x}};$$

$$\therefore x = a\frac{y'^2}{x'^2} \text{ and } y^2 = 4a^2 . \frac{y'^2}{x'^2} \text{ or } y = 2a\frac{y'}{x'};$$

$$\therefore x'^2 + y'^2 = 2a\frac{y'^2}{x'} - a\frac{y'^2}{x'};$$

$$\text{or } y'^2 = \frac{x'^3}{a - x'}, \text{ the equation to the Cissoid of Diocles.}$$

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But if the perpendicular be let fall from the focus, the equation to the parabola reckoning from that point is

$$y^2 = 4a(x+a); \quad \therefore \tan\alpha = \sqrt{\frac{a}{x+a}} = \frac{x'}{y'};$$

$$\therefore x = a\left(\frac{y'^2}{x'^2} - 1\right), \quad y = 2a\frac{y'}{x'};$$

$$\therefore x'^2 + y'^2 = \frac{a}{x'}(x'^2 + y'^2);$$

$$\therefore x' = a;$$

the equation to the line touching the parabola at its vertex.

Ex. 2. If the curve be an ellipse, and the perpendicular be let fall from the center, the equation to the locus of the foot of the perpendicular will be found to be

$$(x^2+y^2)^2 = a^2x^2 + b^2y^2,$$

which agrees with the polar equation already found (Art. 136)

$$r^2 = a^2(1 - e^2\sin^2\theta).$$

In the case of the hyperbola, changing b^2 into $-b^2$, the equation is

$$(x^2+y^2)^2 = a^2x^2 - b^2y^2,$$

which if $b = a$ becomes

$$(x^2+y^2)^2 = a^2(x^2 - y^2),$$

representing a curve called the Lemniscata of Bernouilli, and whose polar equation is $r^2 = a^2\cos 2\theta$.

(8) If a curve roll upon an equal one, similar points being always in contact, to find the locus of any given point in the rolling curve.

Let AP be the fixed, and $A'P$ the rolling curve (fig. 71), S' the describing point, and S a point similarly situated in the fixed curve. Join SS' meeting the common tangent at P in Y; also join SP, $S'P$. Then because the points

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in contact are similar, SP, $S'P$ are equal and equally inclined to the common tangent PY; therefore PY bisects SS' at right angles, and therefore the locus of S' is similar to that of Y, the foot of the perpendicular from S upon the tangent to the fixed curve.

If therefore $y = f(x)$ be the equation to the locus of Y, $\frac{y}{2} = f\left(\frac{x}{2}\right)$ is the equation to the locus of S'. Hence if the curves are equal parabolas, and S the focus, its locus will be a straight line; if S be the vertex, its locus will be the cissoid of Diocles; if the curves are ellipses, and S the focus, its locus will be a circle; if S be the center, the equation to its locus will be

$$x^2 + y^2 = 2\sqrt{a^2x^2 + b^2y^2}.$$

(9) To find the locus of the intersection of a normal to an ellipse and the perpendicular upon it from the center.

Let x', y' be the co-ordinates of P (fig. 46); then the equations to PF, CF are respectively

$$y - y' = \frac{a^2}{b^2}\frac{y'}{x'}(x - x'),$$

$$y = -\frac{b^2}{a^2}\frac{x'}{y'} \cdot x,$$

$$\text{also } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

and it remains to eliminate x' and y' between these equations. By means of the second the first is reduced to

$$y - y' = -\frac{x}{y}(x - x') \quad \text{or } x^2 + y^2 = xx' + yy' = xx'\left(1 - \frac{b^2}{a^2}\right),$$

$$\frac{x'}{a} = \frac{a}{x} \cdot \frac{x^2 + y^2}{a^2 - b^2};$$

$$\therefore \frac{y'}{b} = -\frac{b}{y} \cdot \frac{x^2 + y^2}{a^2 - b^2};$$

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$$\therefore \left(\frac{a^2}{x^2}+\frac{b^2}{y^2}\right)\left(\frac{x^2+y^2}{a^2-b^2}\right)^2=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1,$$

and the polar equation is $r=\dfrac{ae^2\sin\theta\cos\theta}{\sqrt{1-e^2\cos^2\theta}}$.

(10) To find the locus of the intersection of two tangents to an ellipse inclined to one another at a constant angle.

Let PW, $P'W$ be the tangents (fig. 44), CQ, CQ' perpendiculars upon them, $CW=r$, $TCW=\theta$, $QCQ'=2\alpha$; also let $TCQ=\phi-\alpha$, and consequently $TCQ'=\phi+\alpha$, then (Art. 136) supposing $AC=1$,

$$r^2\cos^2(\phi+\alpha-\theta)=1-e^2\sin^2(\phi+\alpha),$$
$$r^2\cos^2(\phi-\alpha-\theta)=1-e^2\sin^2(\phi-\alpha);$$

subtracting these we get

$$r^2\sin 2(\phi-\theta)=e^2\sin 2\phi, \text{ or } \tan 2\phi=\frac{r^2\sin 2\theta}{r^2\cos 2\theta-e^2},$$

by means of which eliminating ϕ from either of the above equations, we find

$$\tan 2\alpha\,(r^2+e^2-2)=2\sqrt{r^2(1-e^2\sin^2\theta)+e^2-1};$$

or in rectangular co-ordinates, putting $\tan 2\alpha=m$,

$$m(x^2+y^2-a^2-b^2)=2\sqrt{a^2y^2+b^2x^2-a^2b^2}.$$

(11) To find the locus of the intersection of two normals to an ellipse at right angles to one another.

Let PF, $P'F'$ be the normals (fig. 43), CF, CF' perpendiculars upon them, $CO=r$, $BCO=\theta$, $BCF=\phi$, $BCF'=\dfrac{\pi}{2}-\phi$, $AC=1$; then

$$r\cos(\phi-\theta)=CF=\frac{e^2\cos\phi\sin\phi}{\sqrt{1-e^2\sin^2\phi}},$$

$$r\sin(\phi-\theta)=CF'=\frac{e^2\cos\phi\sin\phi}{\sqrt{1-e^2\cos^2\phi}},$$

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adding the squares of these equations, we get

$$r^2 = \frac{e^4 (2 - e^2) \tan^2 2\phi}{(2 - e^2)^2 (1 + \tan^2 2\phi) - e^4},$$

or $\tan 2\phi = \frac{2n}{1 + n^2} \frac{r}{\sqrt{c^2 - r^2}}$, putting $n = \frac{1}{b}$, $c = \frac{1 - b^2}{\sqrt{1 + b^2}}$.

Again, dividing one equation by the other and reducing,

$$\cos 2(\phi - \theta) = \frac{1 - n^2}{1 + n^2} \cos 2\phi.$$

Hence substituting the above value of $\tan 2\phi$, we get

$$r = c \frac{\tan^2 \theta - n^2}{\tan^2 \theta + n^2}.$$

(12) To find the locus of the intersection of two normals to a parabola at right angles to one another.

If $\tan PGN = m$ (fig. 29), then $PN = 2am$ and $AN = am^2$; therefore the equation to PG is

$$y - 2am = -m(x - am^2),$$

and changing m into $-\frac{1}{m}$, the equation to a second normal perpendicular to PG is

$$y + \frac{2a}{m} = \frac{1}{m}\left(x - \frac{a}{m^2}\right).$$

Hence, adding and subtracting, we get

$$y = a\left(m - \frac{1}{m}\right),$$

$$x - 3a = a\left(m - \frac{1}{m}\right)^2;$$

$$\therefore\ y^2 = a(x - 3a),$$

the equation to the required locus, which is consequently a parabola.

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(13) To find the locus of the intersection of two tangents to a parabola which include a given angle (fig. 29).

PY, *py* two tangents intersecting at an angle $YQy = YSy = a$, $SQ = r$, $ASQ = \theta$.

$$\text{Now } ASY = QYy = QSy = \theta - a - ASY;$$

$$\therefore ASY = \tfrac{1}{2}(\theta - a), \quad \text{and } QSY = \tfrac{1}{2}(\theta + a);$$

$$\therefore r \cos \tfrac{1}{2}(\theta + a) = SY = a \sec \tfrac{1}{2}(\theta - a);$$

$$\therefore r = \frac{2a}{\cos a + \cos \theta},$$

the equation to a conic section with focus *S*.

(14) To find the locus of the point which is the intersection of three normals to an ellipse.

The equation to the normal at a point (x', y') of an ellipse is

$$y - y' = \frac{a^2}{b^2}\frac{y'}{x'}(x - x'),$$

$$\text{or } yx'\sqrt{1 - e^2} = (x - e^2x')\sqrt{a^2 - x'^2}.$$

Let h, k be co-ordinates of a given point through which the normal passes, then

$$k^2x'^2(1 - e^2) = (h - e^2x')^2(a^2 - x'^2) \ldots\ldots\ldots (1)$$

is the equation for determining x', the abscissa of the point in the ellipse; and as this equation is of the form

$$e^4x'^4 - \&c. - a^2h^2 = 0,$$

it has two or four possible roots; and consequently through the point (h, k) in general either four or two normals can be drawn. If two of the possible roots become equal (which can only happen in the case of four real roots), then three normals will pass through the point (h, k); in that case the derived equation

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$$k^2x'(1-e^2) = -e^2(h-e^2x')(a^2-x'^2) - x'(h-e^2x')^2$$

has one of them. Dividing this by (1), we find

$$\frac{1}{x'} = -\frac{e^2}{h-e^2x'} - \frac{x'}{a^2-x'^2}, \quad \text{or } ha^2 = e^2x'^3;$$

$\therefore\ x' = \left(\frac{ha^2}{e^2}\right)^{\frac{1}{3}}$ satisfies equation (1), and substituting we get

$$k^{\frac{2}{3}}(1-e^2)^{\frac{1}{3}} + h^{\frac{2}{3}} = (ae^2)^{\frac{2}{3}}$$

for the equation of condition that (1) may have equal roots, and as often as h and k satisfy this equation, three normals to the ellipse will pass through the point (h, k). The above is consequently the equation to the locus of the intersection of three normals to an ellipse; and coincides, as might have been foreseen, with the equation to the evolute.

On tracing curves from their equations.

241. When a curve passes through the origin, the angle at which it cuts the axis of x may be determined by taking the limit of $\frac{y}{x}$ when $x = 0$, which will be the value of the tangent of that angle.

Let AP be a curve passing through the origin A (fig. 72), P a point in it near A with co-ordinates $AN = x$, $NP = y$; draw the secant AP, then $\tan PAN = \frac{y}{x}$; now let P move up to and coincide with A, then the secant AP coincides with AT the tangent to the curve at A, and

$$\tan TAN = \text{limit of } \tan PAN = \text{limit of } \frac{y}{x}, \text{ when } x = 0.$$

Hence the angles at which a proposed curve cuts the co-ordinate axes may always be determined; for we have only to transfer the origin to one of the points in question, and in the transformed equation take the limit of $\frac{y}{x}$ by putting $x = 0$.

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242. In tracing curves from their equations, whenever y is given or can be found in an explicit function of x, it will be best to use algebraical processes alone.

First, determine the points where the curve cuts the axis of x; transfer the origin of co-ordinates, if necessary, to one of these points, expand y in a series ascending by powers of x, and let

$$y = ax^m + bx^n + \&c.$$

If $m < 1$, limit of $\frac{y}{x} = \infty$; and therefore the curve is perpendicular to the axis of x, and is concave.

If $m = 1$, limit of $\frac{y}{x} = a$; therefore the curve cuts the axis at an angle $= \tan^{-1}(a)$, and is situated above or below the tangent, i. e. is convex or concave, according as b is positive or negative.

If $m > 1$, limit of $\frac{y}{x} = 0$, and the curve touches the axis of x, and is convex.

Similarly the form of the curve at all its other intersections with both the axes may be found.

Secondly, determine the nature of the infinite branches, and to that end expand y in a descending series of powers of x (on the supposition that both x and y are very great), and let

$$y = ax^m + bx^n + \ldots\ldots + ex + f + \frac{g}{x} + \ldots\ldots$$

$\therefore$ $y = ax^m + bx^n + \ldots\ldots + ex + f$ is the equation to the asymptotic curve, above or below which the given curve is situated, according as g is positive or negative.

If $m = 1$, the equation to the asymptote is $y = ex + f$, a straight line; and the curve will be situated above or below the asymptote, i. e. will be convex or concave to the axis of x, according as g is positive or negative.

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In this case the infinite branch represented by

$$y = ex + f + \frac{g}{x} + \ldots\ldots$$

is said to be hyperbolic.

If $m > 1$, the infinite branch is parabolic, and is concave or convex, according as its asymptote is concave or convex.

Having thus found the figure of the curve at the points where it cuts the axes, and also when x and y are very great, it is easy to trace the intermediate parts. For the actual position of the maximum or minimum ordinates and points of contrary flexure, recourse must be had to the methods of the Differential Calculus.

If the equation to a curve can be resolved, and give for y the values V, V', &c. functions of x and constants, we must trace separately each of the curves represented by $y = V$, $y = V'$, &c., all of which will be particular branches of the proposed curve. The branches cannot terminate abruptly; they will either go on to infinity, or the ordinates will become imaginary, in which case two branches will be united and mutually continue one another.

Ex. 1. To trace the curve whose equation is $ay^3 = (a^2 - x^2)^2$ (fig. 73).

Since the equation does not alter when $-x$ is written for x, the curve is symmetrical with respect to the axis of y; also for any value of x, y has only one possible value, and is always positive. When $x = 0$, $y = a$, and as x increases y diminishes; therefore a is a maximum value of y, and the curve cuts the axis of y at B at right angles, and is concave to the axis of x. When $x = a$, $y = 0$, and the curve cuts the axis of x at right angles at A, because if we remove the origin to that point by making $x = a + x'$, we get $ay^3 = (2ax' + x'^2)^2$, and therefore limit of $\frac{y^3}{x'^3}$ = limit of $\frac{(2a + x')^2}{ax'} = \infty$. When $x > a$, the

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equation becomes $ay^3 = (x^2 - a^2)^2$, and as x increases, y increases, till x is very large, when the relation between them approximates to

$$a^{\frac{1}{3}}y = x^{\frac{4}{3}}\left(1 - \frac{a^2}{x^2}\right)^{\frac{2}{3}} = x^{\frac{4}{3}}\left(1 - \frac{2}{3}\frac{a^2}{x^2}\right) = x^{\frac{4}{3}} - \frac{2}{3}\frac{a^2}{x^{\frac{2}{3}}};$$

$\therefore a^{\frac{1}{3}}y = x^{\frac{4}{3}}$ is the equation to the asymptotic parabola ZOZ', below which the curve lies, because the second term of the expansion of y is negative. Hence the figure of the curve is that annexed, having a point of inflexion at P; for the curve is concave to the axis of x at A, and afterwards convex because the parabola with which it tends to coincide is so. There is of course another point of inflexion at P'.

Ex. 2. To trace the curve whose equation is

$$y = \frac{x - 3}{(x - 1)(x - 2)} \quad \text{(fig. 74)}.$$

When $x = 0$, $y = -\frac{3}{2}$, and the curve cuts the axis of y at D; as long as $x < 1$, y is negative and becomes infinite when $x = 1$; therefore the ordinate BE corresponding to $x = 1$ is an asymptote, and we thus get the branch DE.

When x is between 1 and 2, y is positive, and becomes very great both when x is a little less than 2 and a little greater than 1, and gives the portion $E'GF'$.

When $x = 2$, y is infinite, so that the ordinate FF' is an asymptote.

When x lies between 2 and 3, y is negative and gives the portion FH.

When $x = 3$, $y = 0$, and the curve cuts the axis at H.

When $x > 3$, y is positive and gives the portion HK; and when x is very great,

$$y = \frac{x}{x^2 - 3x} = \frac{1}{x - 3} = 0;$$

therefore the axis of x is an asymptote.

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When x is negative, the equation is

$$y = -\frac{x+3}{(x+1)(x+2)};$$

therefore y is always negative, and diminishes as x increases, and becomes 0 when $x = \infty$ and gives the branch DL.

Ex. 3. To trace the curve whose equation is

$$ay = x^2 + x\sqrt{2ax - x^2} \quad \text{(fig. 75)}.$$

Taking the radical, which of course admits of a double sign, first with a positive sign, we have when $x = 0$, $y = 0$, and limit of $\frac{y}{x} = 0$; therefore the curve passes through the origin A and touches the axis of x; when $x = a$, $y = 2a$, and when $x = 2a$, $y = 4a$, beyond which y is impossible; we thus get the portion of the curve AED, the ordinate CD being a tangent at D. Again taking the radical with a negative sign, so that

$$ay = x^2 - x\sqrt{2ax - x^2},$$

$x = 0$ gives $y = 0$, and limit of $\frac{y}{x} = 0$, as before; as long as $x < a$, y is negative; when $x = a$, $y = 0$, and putting $x' + a$ for x we get, supposing x' very small,

$$ay = ax' + \frac{3}{2}x'^2; \qquad \therefore \text{ limit of } \frac{y}{x'} = 1;$$

hence the portion of the curve AFB touches the axis of x at A, and cuts it at an angle of 45^0 at B, and is there situated above the tangent. When $x = 2a$, $y = 4a = DC$, and for greater values of x, y is impossible; hence DC is a tangent to both portions of the curve at D, the point in which they are united. For all negative values of x, y is impossible. The curve is therefore such as is represented, the tangent being parallel to the axis of x at F and E.

THE END.

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A TREATISE

ON

ANALYTICAL GEOMETRY

OF

THREE DIMENSIONS,

CONTAINING THE

THEORY OF CURVE SURFACES,

AND OF

CURVES OF DOUBLE CURVATURE.

BY JOHN HYMERS, M.A.

FELLOW AND TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

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ANALYTICAL GEOMETRY

OF

THREE DIMENSIONS.

SECTION I.

ON THE PLANE, AND STRAIGHT LINE.

1. In order to determine the position of a point in space, some fixed point is taken for the *origin of co-ordinates*, and through it are drawn three fixed planes, called the *co-ordinate planes*, at right angles to one another, and intersecting one another two and two in straight lines, which are also at right angles to one another, and are called the *axes of the co-ordinates*. Then, if the perpendicular distances of a point from each of the co-ordinate planes be given, its position will be completely determined. For let O (fig. 1.) be the origin of the co-ordinates, yOz, zOx, xOy the co-ordinate planes; M any point, and ME, MF, MN the perpendiculars let fall from it upon the co-ordinate planes; these perpendiculars are called the *rectangular co-ordinates* of M, and, as their values change for the different points of space, are denoted by the variables x, y, z. Then the point M will be determined in position, if we know the values of its three co-ordinates; that is, if we know that for that point

$$x = a, \quad y = b, \quad z = c.$$

For, if along Ox we measure $OA = a$, and through A draw an indefinite plane parallel to yOz, this plane will contain all points whose distance from yOz is a, or for which $x = a$,

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and, therefore, the point in question. Similarly, if we measure along Oy, Oz, the distances $OB = b$, $OC = c$, and through B, C, draw indefinite planes respectively parallel to zOx, xOy, each of these must contain the point in question; therefore the three planes will, by their intersection in M, determine one single position for the point whose co-ordinates are $x = a$, $y = b$, $z = c$; which position coincides, as we see, with the angular point opposite to the origin of the rectangular parallelopiped constructed upon the edges OA, OB, OC, equal to the three given co-ordinates.

2. The feet of the perpendiculars E, F, N, are called the *projections* of the point M upon the co-ordinate planes. Of these if any two be given, the third can be found; thus if E, F be given, draw EB, FA, parallel to Oz; then BN, AN, drawn respectively parallel to Ox, Oy, will by their concourse determine N.

Instead of the perpendiculars ME, MF, MN, the three lines OA, AN, NM, which are respectively equal to them, are commonly called the co-ordinates of M, and denoted by x, y, z; also the axes of the co-ordinates Ox, Oy, Oz, are often called the axis of x, the axis of y, and the axis of z; and the co-ordinate planes xOy, yOz, zOx, the planes of xy, yz, zx, respectively.

3. The determination of the point M will not however be complete, unless we take into account the *signs* of the quantities a, b, c, in the equations

$$x = a, \quad y = b, \quad z = c;$$

in order to measure these distances, when they are positive, along the positive parts Ox, Oy, Oz, of the co-ordinate axes; or along the negative parts Ox', Oy', Oz', of the axes produced in the contrary direction, when they are negative; as is explained in Plane Geometry. For since the co-ordinate planes, which must be supposed to be prolonged indefinitely, form among themselves eight solid angles, there are eight positions in which M might be situated at absolute

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distances a, b, c from the co-ordinate planes; and it is only attention to the signs which will enable us to select the true one. Without going through the combination of signs which belongs to each of the solid angles, it may be sufficient to observe, that if a point be situated in the compartment xyz, all its co-ordinates will be positive; if in the compartment $x'y'z'$, all negative; and that for points in the compartments $xy'z$, $x'yz'$, we must have

$$x = a, \quad y = -b, \quad z = c; \quad x = -a, \quad y = b, \quad z = -c,$$

respectively.

4. Sometimes it is requisite to take the co-ordinate planes not at right angles, but inclined at given angles to one another, in which case the system of co-ordinates is called *oblique*. Thus (fig. 2.) if xOy, yOz, zOx be three planes drawn through the point O, and intersecting one another two and two at given angles, in the lines Ox, Oy, Oz; and if from any point M, MN be drawn parallel to Oz meeting the plane xOy in N, and NA be drawn in that plane parallel to Oy; OA, AN, NM, are the co-ordinates of the point M referred to the oblique axes Ox, Oy, Oz; and N is called the projection of M made parallel to the axis Oz.

In the present chapter we shall confine our attention to the case of *rectangular* co-ordinates; noticing however, as they arise, several relations of lines and planes to one another, the analytical expressions of which, and mode of investigation, are the same, whether the axes of the co-ordinates be rectangular or oblique.

5. To find the distance of a point from the origin in terms of its co-ordinates.

Let M (fig. 1.) be the point,

$OA = x'$, $AN = y'$, $NM = z'$ its given co-ordinates;

join OM, ON, and let $OM = d$.

Then from the triangle ONM, right-angled at N,

$$OM^2 = ON^2 + NM^2;$$

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but the triangle OAN, right-angled at A, gives

$$ON^2 = OA^2 + AN^2;$$

$$\therefore\ OM^2 = OA^2 + AN^2 + NM^2;$$

$$\therefore\ d = \sqrt{x'^2 + y'^2 + z'^2}.$$

Cor. Let α, β, γ be the angles which OM makes with the axes of x, y, z; then, joining AM, since angle OAM is a right angle,

$$\cos\alpha = \frac{OA}{OM} = \frac{x'}{d};$$

$$\therefore\ x' = d\cos\alpha;\quad \text{similarly, } y' = d\cos\beta,\quad z' = d\cos\gamma.$$

$$\text{Also } (\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = \frac{x'^2 + y'^2 + z'^2}{d^2} = 1.$$

This is the condition which the three angles, made by a line through the origin with the co-ordinate axes, must satisfy; they cannot, therefore, be assumed at pleasure, but two being given, α, β, for instance, the third is determined by the equation

$$\cos\gamma = \pm\sqrt{1 - (\cos\alpha)^2 - (\cos\beta)^2},$$

and will therefore have two values, γ and $\pi - \gamma$; the value $\pi - \gamma$ corresponding to a line OM' in the plane zOM, which is inclined at an angle γ to Oz', and visibly makes angles equal to α, β with Ox, Oy, the same as OM does.

6. To find the distance between two points in terms of their co-ordinates.

Let M' (fig. 3.) be a point whose co-ordinates are x', y', z', and M any other point whose co-ordinates are x, y, z; join $MM' = d$, and upon MM', as diagonal, describe a rectangular parallelopiped having its edges parallel to the co-ordinate axes.

Then from the triangle $M'MK$, right-angled at K,

$$MM'^2 = M'K^2 + KM^2 = M'K^2 + (z - z')^2.$$

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But the triangle $N'LN$, right-angled at L, gives

$$M'K^2 = N'N^2 = (x - x')^2 + (y - y')^2;$$

$$\therefore\ MM'^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

$$\text{or}\quad d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

Both in this formula, and in that of Art. 5, we take the radical with a positive sign; as the question only relates to the absolute distance of the points.

Cor. Let α, β, γ be the angles which MM' makes with the edges of the parallelopiped, and which are manifestly the same as those made by a line OP drawn through the origin parallel to MM', with the co-ordinate axes. Then from the triangle $MM'H$, right-angled at H, we have

$$\cos\alpha = \frac{M'H}{MM'} = \frac{x - x'}{d};$$

$$\text{similarly,}\ \cos\beta = \frac{y - y'}{d},\quad \cos\gamma = \frac{z - z'}{d}.$$

7. To find the equation to a plane.

A plane is a surface in which if any point be taken, it is equidistant from two given points.

Let one of the given points O (fig. 4.) be the origin, and let x', y', z', denote the co-ordinates of the other given point M; and let P be a point in the plane of which the co-ordinates are x, y, z;

$$\therefore\ OP^2 = x^2 + y^2 + z^2,$$

$$\text{and}\ \ PM^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2;$$

and these by the definition are equal,

$$\therefore\ x^2 + y^2 + z^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2;$$

or, expanding and reducing,

$$2(xx' + yy' + zz') = x'^2 + y'^2 + z'^2,\ (1),$$

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a relation among the co-ordinates of any point in the plane, and therefore its equation; which, since x', y', z' are constant quantities, is of the form $z = Ax + By + c$.

Cor. 1. Join OM, meeting the plane in Q, and join PQ. Then $OQ = QM$, since Q is a point in the plane; also $OP = PM$, and therefore $\angle OQP = \angle PQM$, or OM is perpendicular to PQ; and PQ is any line in the plane, therefore OM is perpendicular to the plane, and is bisected in the point Q in which it meets the plane. Let $OQ = d$, and let α, β, γ, be the angles which it makes with the axes of x, y, z; then (Cor. Art. 5.)

$$x' = 2d\cos\alpha, \quad y' = 2d\cos\beta, \quad z' = 2d\cos\gamma;$$

therefore, substituting in equation (1),

$$4d\,(x\cos\alpha + y\cos\beta + z\cos\gamma) = 4d^2,$$

or $$x\cos\alpha + y\cos\beta + z\cos\gamma = d,$$

another form of the equation to a plane, in terms of the perpendicular upon it from the origin, and the angles which the perpendicular makes with the co-ordinate axes produced in the positive direction.

Cor. 2. Let a, b, c, denote the portions of the co-ordinate axes OA, OB, OC, intercepted between the origin and the plane; join QA, then from the triangle OQA, right-angled at Q, we have $\cos\alpha = \frac{OQ}{OA} = \frac{d}{a}$; similarly, $\cos\beta = \frac{d}{b}$, $\cos\gamma = \frac{d}{c}$; therefore, substituting in the equation of Cor. 1, the equation to the plane may be put also under the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

8. We may also consider a plane as a surface generated by a straight line which moves, so as always to remain parallel to itself, and to intersect a straight line given in position.

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Let AC (fig. 5.) in the plane of zx be the line to which the generating line PQ is always parallel, and BC in the plane of yz, intersecting the former in C, the line along which it moves. Let $z = Ax + c$ be the equation to the line AC referred to the axes Ox, Oz, in the plane in which it lies; similarly, let $z = By + c$ be the equation to BC referred to the axes Oy, Oz, where in both cases $c = OC$. From P any point in the generating line, and from Q where it meets BC, draw PN, QM, parallel to the axis of z, and join MN. Then the plane QN is parallel to zx, and MN is parallel to Ox; and therefore PQ is inclined to MN at the same angle that AC is to Ox; and since MN, NP, are the co-ordinates of P in the plane QN,

$$PN = A \,.\, MN + QM.$$

But, Q being a point in the line BC, $QM = B \,.\, MO + OC$,

$$\therefore\ PN = A \,.\, MN + B \,.\, MO + OC;$$

or, if x, y, z denote the co-ordinates of P,

$$z = Ax + By + c,$$

the equation to the plane.

Cor. The lines in which a plane intersects the co-ordinate planes are called its *traces* on those planes; thus AC, BC, are two of the traces of the plane BCA, the third being the line in which, if prolonged, it would intersect the plane of xy. Hence, we perceive the meaning of the constants in the equation to a plane, 1st p. 11.

$$z = Ax + By + c;$$

for A, B, are the tangents of the angles at which the traces on zx, yz, are respectively inclined to the axes of x and y produced in the positive directions; and c is the portion of the axis of z produced in the positive direction, intercepted between the plane and the origin.

9. It is important to observe, that the foregoing method of finding the equation to a plane, applies equally to the case where the co-ordinates are oblique, and leads

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to a result of the same form; so that whether a plane be referred to rectangular or oblique axes, its equation may be represented by $z = Ax + By + c$; but in the latter case, A will signify the *ratio of the sines* of the angles at which the trace on zx is inclined to the axes of x and z; and B the *ratio of the sines* of the angles at which the trace on yz is inclined to the axes of y and z. Hence, all results which involve no other assumption than $z = Ax + By + c$, for the form of the equation to a plane, will be equally true for oblique and rectangular co-ordinates.

10. The particular cases of the equation to a plane, when it involves only one, or only two of the variables, require a separate notice.

Thus the equation $x = a$, since it belongs to all points whose distance from the plane of yz, measured parallel to the axis of x, is a, represents an indefinite plane parallel to yz. Similarly, the equation $x = 0$, characterizes all points situated in the plane of yz, or is the equation to that plane; and $y = 0$, $z = 0$, are respectively the equations to the planes of zx, xy.

Again, the equation $my = nx + b'$ represents not only the line AB (fig. 6.) in the plane of xy, but all points situated in the plane RA drawn through AB parallel to the axis of z; for any one of these points P, whatever be the value of PM or z, will have the same x and y as its projection M, and therefore its co-ordinates will satisfy the equation $my = nx + b'$ which does not involve z. Similarly, the equations $x = mz + a$, $y = nz + b$, represent planes parallel to the axes of y and x; *i. e.* each equation represents a plane parallel to that axis whose co-ordinate it does not involve. This also appears from the equation $z = Ax + By + c$; for if the plane be parallel to the axis of y, then its trace on yz will be parallel to the axis of y, and therefore $B = 0$, or $z = Ax + c$ is its equation, the same as the equation to its trace on yz; similarly, if the plane be parallel to the axis of x, $A = 0$. The equation requires to be modified in order to deduce from it the equation to a plane parallel to the axis of z; dividing

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by c, and observing (fig. 4.) that $A = -\frac{OC}{OA}$, or $\frac{A}{c} = -\frac{1}{a}$, and similarly $\frac{B}{c} = -\frac{1}{b}$, the equation becomes $\frac{z}{c} = -\frac{x}{a} - \frac{y}{b} + 1$; now let c be infinite, then the plane becomes parallel to the axis of z, and its equation is $\frac{x}{a} + \frac{y}{b} = 1$, the same as that to its trace AB. If the plane pass through the origin, $c = 0$, and its equation is $z = Ax + By$.

11. Every equation of the first order involving three variables is the equation to a plane. Let the equation be reduced to the form $z = Ax + By + c$, and to find the trace of the surface which it represents on yz, let $x = 0$; $\therefore z = By + c$, which is the equation to a straight line; let this be CB (fig. 5). Next to determine the sections made by planes parallel to zx, let $y = b$, b', &c.; $\therefore z = Ax + Bb + c$, $z = Ax + Bb' + c$, &c., which represent straight lines QP, $Q'P'$, &c., parallel to one another; they likewise all intersect the trace CB; for making $x = 0$, we have $Bb + c$, $Bb' + c$, &c., for the values of QM, $Q'M'$, &c., and these are the ordinates of CB corresponding to $y = b$, $y = b'$, &c. Therefore the locus of the proposed equation is a system of parallel straight lines all passing through a fixed line; *i. e.* it is a plane.

12. The equation $z = Ax + By + c$,

where A, B, denote numbers, and c a line, which we shall take for our standard equation to a plane, has the defect of not being symmetrical; to compensate for which, it contains only three constants, each having an obvious signification, so as to enable us readily to interpret any results to which its use may lead; moreover it leads to the analytical expressions of the more useful relations of lines and planes to one another, under forms the simplest possible; and it is readily adapted to planes in all particular positions, with the single exception of a plane parallel to the axis of z. It has the advantage too of being the form under which the general

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equation to the tangent plane of a curve surface presents itself.

13. To find the equations to a straight line in space.

A straight line is the intersection of two planes, and consequently is given when the equations to any two planes which contain it are given.

Among the various planes which may be drawn through a given straight line, those are employed, for the sake of simplicity, to determine it, which are parallel to the co-ordinate axes, since their equations will involve only two of the variables. From all the points of the line MM'' in space (fig. 6.) let perpendiculars MC, $M'C'$, &c., be drawn to the plane of xy, meeting it in the points C, C', &c.; the assemblage of these points is called the *projection* of MM'' on this plane. This projection will manifestly be a straight line, since all the perpendiculars are situated in the same plane, which is parallel to Oz, and is called the *projecting plane* of MM''. Similarly, if we project MM'' upon the other co-ordinate planes by perpendiculars upon them from its various points, we shall have the three projections AA', BB', CC', of which any two are sufficient to determine MM'. For suppose AA' and BB' are given; if through these we draw planes respectively perpendicular to zx, yz, each of them must contain the line MM', and will therefore fix its position in space by their intersection. But the positions of the projections NA, QB in the planes zx, yz, will be determined by their equations $x = mz + a$, $y = nz + b$, in which m and n denote the tangents of the angles at which they are respectively inclined to the axis of z, and $a = ON$, $b = OQ$. Hence the system of equations

$$\left\{ \begin{array}{l} x = mz + a, \\ y = nz + b, \end{array} \right.$$

which belong to the projections of the line, or rather to its projecting planes, will completely determine it.

Cor. 1. Any values of x, y, z which satisfy the above equations are co-ordinates of some point in the line; for suppose x', y', z', were such values, and let M' be the point in

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MP for which $z = z'$; then for each of its projections A', B', $z = z'$, and therefore $x' = ON'$, $y' = OQ'$; that is, if $z' = C'M'$, then x' and y', provided they be determined from the above equations, will equal ON', $N'C'$. Hence the equations $x = mz + a$, $y = nz + b$, may be called the equations to the line MP itself.

Cor. 2. Corresponding to the value $M'C'$ of z, the values of x and y in the simultaneous equations $x = mz + a$, $y = nz + b$, are ON', $N'C'$, the co-ordinates of the point C' of the projection PC; and therefore the equation to the third projection may always be deduced from the two others by eliminating z, which gives

$$\frac{x-a}{m} = \frac{y-b}{n}, \text{ or } nx - my = na - mb.$$

14. In the same manner it might be shewn, that if the axes of the co-ordinates were oblique, the equations to a straight line would still be of the form

$$x = mz + a, \quad y = nz + b;$$

the only difference would be that the projections would be made by planes parallel to the axes of the co-ordinates; and m would denote the ratio of the sines of the angles which the projection on zx makes with the axes of z and x respectively; and similarly for n.

15. The point in which a line meets any one of the co-ordinate planes, is called its *trace* upon the plane. The positions of these points are determined by putting, in the equations to the line, x, y, z, successively $= 0$; thus the co-ordinates of the traces of the line whose equations are

$$x = mz + a, \quad y = nz + b, \quad \frac{x-a}{m} = \frac{y-b}{n},$$

upon the planes of yz, zx, xy, are respectively

$$\left.\begin{array}{l} z = -\dfrac{a}{m} \\ y = -\dfrac{na}{m} + b \end{array}\right\} \quad \left.\begin{array}{l} z = -\dfrac{b}{n} \\ x = -\dfrac{mb}{n} + a \end{array}\right\} \quad \left.\begin{array}{l} x = a \\ y = b \end{array}\right\}.$$

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16. These principles being laid down, we proceed to the resolution of several Problems relative to the plane and straight line, the results of which are of great use. As the equation to a plane contains three disposable constants, a plane may be drawn so as to fulfil various conditions; as, for instance, to pass through three given points; to pass through a given point and be parallel to a plane, or to each of two given straight lines; or to pass through a given point and be perpendicular to a given straight line. Similarly, a straight line, since its two equations contain four disposable constants, may be drawn so as to satisfy various conditions.

17. To express that a plane passes through a given point.

Let $z = Ax + By + c$, be the equation to a plane passing through a point (x', y', z'); then the equation to the plane must be satisfied by the co-ordinates of the point, $\therefore z' = Ax' + By' + c$; therefore, subtracting this equation from the former to eliminate c, we have

$$z - z' = A(x - x') + B(y - y'),$$

in which A and B remain undetermined, so that the plane may still be made to satisfy two other conditions.

18. To find the conditions that two planes, whose equations are given, may be parallel to one another.

Let $z = Ax + By + c$, $z = A'x + B'y + c'$, be the equations to two planes; then if they be parallel to one another, their common sections with any plane which cuts them are parallel; therefore their traces on the co-ordinate planes are parallel. But, making $y = 0$, the traces on the plane of zx have for their equations $z = Ax + c$, $z = A'x + c'$, which will be parallel if $A = A'$; similarly, since the traces on yz are parallel, $B = B'$. Hence the conditions of parallelism of two planes are, that the coefficients of x and y in their equations are respectively equal.

Cor. The equation to a plane drawn through a point (x', y', z'), parallel to a given plane $z = A'x + B'y + c'$, is

$$z - z' = A'(x - x') + B'(y - y').$$

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19. To find the equation to a plane which shall pass through three given points.

Let $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, be the co-ordinates of the given points, and $z = Ax + By + c$ the equation to the plane; then the co-ordinates of the points must satisfy it;

$$\therefore\ z_1 = Ax_1 + By_1 + c,$$

$$z_2 = Ax_2 + By_2 + c,$$

$$z_3 = Ax_3 + By_3 + c,$$

which are the three equations for finding A, B, c. Multiply the second and third respectively by indeterminate coefficients t, u, and add them to the first;

$$\therefore\ z_1 + tz_2 + uz_3 = A(x_1 + tx_2 + ux_3).....(1),$$

$$\text{provided } y_1 + ty_2 + uy_3 = 0, \quad 1 + t + u = 0.$$

These two latter equations give

$$y_1 - y_3 + t(y_2 - y_3) = 0, \quad y_1 - y_2 + u(y_3 - y_2) = 0,$$

and so determine t and u; and substituting these values in equation (1), we find A; and then B may be obtained by interchanging $x_1x_2x_3$ and $y_1y_2y_3$; so that

$$z - z_1 = A(x - x_1) + B(y - y_1),$$

the equation to the plane, will be completely determined.

20. To determine the line of intersection of two planes whose equations are given.

Let the equations to the planes be

$$z = Ax + By + c, \quad z = A'x + B'y + c'.$$

Then the line of intersection is, as we know, sufficiently determined by these two equations taken simultaneously; that is, by supposing x, y, z to receive only such values as satisfy both at the same time. If, for instance, we substitute any series of values for z, we shall have for each of these values two equations to find the corresponding

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values of x and y; and thus we may determine as many points in the line as we please. But the line of intersection is more conveniently determined by its projections on the co-ordinate planes. To find these projections, let $x'y'z'$ denote the co-ordinates of any point in the intersection, then x', z', are also the co-ordinates of the corresponding point of the projection on zx;

$$\therefore\ z' = Ax' + By' + c, \quad z' = A'x' + B'y' + c';$$

$\therefore$, eliminating y', $(B' - B)z' = (AB' - A'B)x' + B'c - Bc'$;

which is the relation between the co-ordinates of any point in the projection on zx. Hence, suppressing the accents,

$(B' - B)z = (AB' - A'B)x + B'c - Bc'$, and, similarly,

$(A' - A)z = (BA' - B'A)y + A'c - Ac'$,

are the equations to the line of intersection of the planes; which, as we see, result from separately eliminating y and x from the equations to the planes.

21. To find the equations to a straight line which shall pass through two given points.

Let $x'y'z'$, $x''y''z''$, be the co-ordinates of the given points;

$x = mz + a$, $y = nz + b$, the equations to the line,

which must be satisfied by the co-ordinates of the points;

$$\therefore\ x' = mz' + a, \quad y' = nz' + b \ldots\ldots (1),$$
$$x'' = mz'' + a, \quad y'' = nz'' + b,$$

the four equations to determine the four constants. First, for the values of m and n, we have

$$x' - x'' = m(z' - z''), \quad y' - y'' = n(z' - z'');$$

also subtracting equations (1) from the assumed equations in order to eliminate a and b,

$$x - x' = m(z - z'), \quad y - y' = n(z - z');$$

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therefore, substituting for m and n, the required equations are

$$x - x' = \frac{x' - x''}{z' - z''}(z - z'), \quad y - y' = \frac{y' - y''}{z' - z''}(z - z').$$

Cor. If the line is required to pass through only one point $x'y'z'$, its equations will be

$$x - x' = m(z - z'), \quad y - y' = n(z - z'),$$

where m and n remain undetermined, as might have been foreseen; for this condition may be satisfied by an unlimited number of lines. If, besides, it is to be parallel to a known line of which the equations are

$$x = m'z + a', \quad y = n'z + b',$$

since the projecting planes and therefore the projections of these lines must be parallel, we have $m = m'$, $n = n'$; therefore the required equations are

$$x - x' = m'(z - z'), \quad y - y' = n'(z - z').$$

22. To determine the point of intersection of two given lines.

Let their equations be

$$\left.\begin{aligned} x &= mz + a \\ y &= nz + b \end{aligned}\right\}, \quad \left.\begin{aligned} x &= m'z + a' \\ y &= n'z + b' \end{aligned}\right\}.$$

For any given value of z, the values of x and y furnished by these two systems of equations will in general be different; but if z be the ordinate of their point of intersection, the corresponding values of x and y furnished by them, will be the same. Therefore the co-ordinates of the point of intersection are the values of x, y, z, which simultaneously satisfy the above four equations; and as the number of equations exceeds the number of quantities to be determined, there will be an equation of condition, without which the problem would be impossible; since two straight lines do not, in general, intersect. Hence, eliminating x and y separately,

$$0 = (m - m')z + a - a', \quad 0 = (n - n')z + b - b';$$

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and equating these two values of z,

$$\frac{a-a'}{m-m'} = \frac{b-b'}{n-n'},$$

an equation which must be made identically true by the constants entering into the equations to the two lines, in order that they may intersect. The two remaining co-ordinates of the point of intersection, will be found by substituting the value of

$$z = -\frac{a-a'}{m-m'} = -\frac{b-b'}{n-n'},$$

in either of the above systems of equations.

Cor. When $m = m'$, and $n = n'$, the equation of condition is satisfied, and yet the lines do not intersect, being parallel; therefore the above condition will not ensure the intersection of the lines, without the restriction that the value of z is not to be infinite. Taken without any restriction, it expresses that the lines are in the same plane and may therefore intersect; without deciding at what distance that intersection will happen.

23. To find the conditions in order that a straight line and plane whose equations are given, may coincide, or be parallel to one another.

Let $z = Ax + By + c$, be the equation to the plane,

$$x = mz + a, \quad y = nz + b,$$

the equations to a line coinciding with it. Since every point of the line is in the plane, if z be the ordinate of any point, and $\therefore$ $mz + a$, $nz + b$, the values of the other co-ordinates, these values must satisfy the equation to the plane;

$$\therefore\ 0 = (Am + Bn - 1)\, z + Aa + Bb + c;$$

and since this equation must be true for all values of z,

$$\therefore\ Am + Bn - 1 = 0, \quad Aa + Bb + c = 0,$$

which are the required conditions.

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If the plane and line are parallel, then a plane and line drawn through the origin respectively parallel to them, coincide; therefore the above equations must be satisfied when we suppose c, b, a, to vanish; hence the remaining equation

$$Am + Bn - 1 = 0,$$

expresses the condition in order that a line and plane may be parallel to one another.

24. It will be observed that the results of the preceding Articles from (17) inclusive, are also true, when the axes are oblique; since the only assumptions made, are the forms of the equations to the plane and straight line; but in that case, the constants A, B, m, n, will have different significations.

25. To find the conditions that a straight line and plane may be at right angles to one another.

Let the straight line OP (fig. 7.), be a perpendicular from the origin on the plane ABC, whose traces on the co-ordinate planes are CB, BA, AC; and let it meet the plane in P. Join CP and produce it to meet AB in Q, join OQ. Then the plane COQ, since it passes through CO and OP, is perpendicular to each of the planes AOB, ACB; and therefore AB, the intersection of the two latter planes, is perpendicular to the plane COQ, and therefore to the line OQ, which is the projection of OP on the plane AOB. Similarly, if OR, OS be the projections of OP on the other co-ordinate planes, it may be shewn that they are respectively perpendicular to the traces AC, CB. Hence, when a straight line is perpendicular to a plane, the trace of the plane, and the projection of the line on any one of the co-ordinate planes, are at right angles to one another. And, conversely, if any two projections OR, OS, be perpendicular to the traces AC, CB, the line in space will be perpendicular to the plane; for the projecting planes which pass through OR, OS, being respectively perpendicular to AC, BC, lines in the plane ACB,

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are both perpendicular to the plane ACB, and therefore their intersection, which is the line OP, is perpendicular to that plane.

Now let $z = Ax + By + c$ be the equation to a plane,

$$x = mz + a, \quad y = nz + b,$$

the equations to a line perpendicular to it. Then the equation to the trace of the plane on zx is $z = Ax + c$, and this is perpendicular to the projection of the line on zx, whose equation is

$$z = \frac{x}{m} - \frac{a}{m}; \quad \therefore \frac{A}{m} + 1 = 0.$$

Similarly, the lines whose equations are

$$z = By + c, \quad \text{and} \quad y = nz + b,$$

are perpendicular to one another, and $\therefore \frac{B}{n} + 1 = 0.$

Hence $A + m = 0$, $B + n = 0$, are the required conditions.

Cor. Hence we can find the equations to a straight line which shall pass through a given point, and be perpendicular to a given plane.

Let x', y', z' be the co-ordinates of the given point,

$z = Ax + By + c$ the equation to the plane;

then $x - x' = m(z - z')$, $y - y' = n(z - z')$ are the forms of the required equations (Cor. Art. 21); and since the line is perpendicular to the plane,

$$m = -A, \quad n = -B,$$

$$\therefore x - x' + A(z - z') = 0, \quad y - y' + B(z - z') = 0,$$

are the required equations.

26. The only exception which the assertion, that a line is perpendicular to a plane when two projections are perpendicular to two traces, suffers, is when the plane is

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parallel to one of the co-ordinate axes, as the axis of z; for in that case the two projecting planes become coincident, and a line CQ (fig. 6.) may have two of its projections CS, CR perpendicular to AS, BR the parallel traces on zx and yz, without being perpendicular to the plane AR.

Suppose that $y = -\frac{b}{a}x + b$ was the equation to the plane; then every line perpendicular to it would be in a plane parallel to xy, and therefore have $z = c$ for one of its equations, and $y = \frac{a}{b}x + b'$ for the other, since its projection on xy must be perpendicular to the trace on that plane.

27. In fig. 7, $\angle CQO$ is the inclination of the plane ABC to the plane of xy, and is equal to the $\angle COP$, which OP makes with the axis of z, each being equal to the complement of $\angle POQ$; in other words, the angle between two planes is equal to the angle between two lines respectively perpendicular to them. If the perpendiculars should not meet, then by the angle between them, and generally by the angle between any two lines that do not intersect, we understand the angle formed by two lines drawn through any the same point respectively parallel to them.

28. Having given the equations to a straight line, to find the angles which a line parallel to it through the origin, makes with the axes.

Let $x = mz + a$, $y = nz + b$, be the equations to the given line. Through the origin draw a line OA (fig. 8.) parallel to it; then the equations to OA are

$$x = mz, \quad y = nz.$$

Let α, β, γ, denote the angles which OA makes with the axes of x, y, z; also let $OA = 1$, and let x', y', z', be the co-ordinates of the point A, then (Art. 5.) $x' = \cos\alpha$, $y' = \cos\beta$, $z' = \cos\gamma$; also since the point is in OA,

$$x' = mz', \quad y' = nz'.$$

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But $OA^2 = x'^2 + y'^2 + z'^2$, or $1 = (m^2 + n^2 + 1)z'^2$,

$$\therefore z' = \frac{1}{\sqrt{1 + m^2 + n^2}}; \quad \therefore \cos\gamma = \frac{1}{\sqrt{1 + m^2 + n^2}},$$

$$\cos\beta = \frac{n}{\sqrt{1 + m^2 + n^2}}, \quad \cos\alpha = \frac{m}{\sqrt{1 + m^2 + n^2}}.$$

The radical in these expressions admits of a double sign + or −; but as it must be taken + in all, or − in all, we have only two systems of values for α, β, γ; viz., the angles which OA makes with Ox, Oy, Oz; or their supplements, which AO produced or OA' makes with the same lines. It is usual to take the positive sign; then $\cos\gamma$ is +, and therefore γ acute; and the angles determined are those which the part of the line above the plane of xy makes with the co-ordinate axes produced in the positive directions.

Cor. Conversely these equations give the values of the coefficients m and n,

$$m = \frac{\cos\alpha}{\cos\gamma}, \quad n = \frac{\cos\beta}{\cos\gamma}.$$

Hence the equations to a line passing through a given point (x', y', z') and making angles α, β, γ with the axes, are

$$\frac{x - x'}{\cos\alpha} = \frac{y - y'}{\cos\beta} = \frac{z - z'}{\cos\gamma}.$$

29. To find the angles of inclination of a plane whose equation is given, to the co-ordinate planes.

Let $z = Ax + By + c$ be the equation to the plane;

then $x = -Az$, $y = -Bz$,

are the equations to a line through the origin perpendicular to it; and if α, β, γ denote the angles which this line makes with the axes of x, y, z, then (Art. 27.) α, β, γ

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are also the angles at which the plane is inclined to the co-ordinate planes of yz, zx, xy respectively; and by the preceding article we have

$$\left.\begin{aligned} \cos\alpha &= \frac{-A}{\sqrt{1+A^2+B^2}}, \quad \cos\beta = \frac{-B}{\sqrt{1+A^2+B^2}}, \\ \cos\gamma &= \frac{1}{\sqrt{1+A^2+B^2}}. \end{aligned}\right\}$$

30. Having given the equations to two straight lines, to find the angle which two lines parallel to them drawn through the origin include.

Let $\left.\begin{matrix} x = mz + a \\ y = nz + b \end{matrix}\right\}$, $\left.\begin{matrix} x = m'z + a' \\ y = n'z + b' \end{matrix}\right\}$ be the equations to the lines to which OA, OB, (fig. 8.) are respectively parallel; then $x = mz$, $y = nz$ are the equations to OA, and $x = m'z$, $y = n'z$, those to OB. Take OA, OB, each $= 1$; and let the co-ordinates of A and B be denoted respectively by x, y, z; x', y', z'; join AB, and let $AB = d$, $\angle AOB = \theta$. Then from the triangle AOB,

$$1^2 + 1^2 - 2\cos\theta = d^2 = (x-x')^2 + (y-y')^2 + (z-z')^2;$$

or, expanding, and reducing by the relations

$$x^2 + y^2 + z^2 = 1, \quad x'^2 + y'^2 + z'^2 = 1,$$

$$\cos\theta = xx' + yy' + zz'.$$

But since A and B are situated respectively in OA, OB,

$$x = mz, \quad y = nz; \quad x' = m'z', \quad y' = n'z';$$

$$\therefore\ 1 = (m^2 + n^2 + 1)z^2, \quad \text{or} \quad z = \frac{1}{\sqrt{1+m^2+n^2}};$$

$$\text{Similarly,} \quad z' = \frac{1}{\sqrt{1+m'^2+n'^2}};$$

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$$\therefore \cos\theta = (mm' + nn' + 1)zz' = \frac{mm' + nn' + 1}{\sqrt{1 + m^2 + n^2}\sqrt{1 + m'^2 + n'^2}}.$$

This expression, on account of the two radicals it contains, will furnish for θ, four values equal two and two, corresponding to the four angles formed by the indefinite lines AA', BB'. It is usual to take the denominator positive, *i. e.* to take the radicals with the same sign, so that the points A, B, are both above or both below the plane of xy; and the angle determined is AOB or its opposite $A'OB'$, contained between the two portions of the lines which form each an acute angle with Oz or Oz'; this angle, moreover, will be acute or obtuse according as $1 + mm' + nn'$ is positive or negative.

Cor. By means of the formula $\sin = \sqrt{1 - (\cos)^2}$ we may shew that

$$\sin\theta = \frac{\sqrt{(m - m')^2 + (n - n')^2 + (mn' - m'n)^2}}{\sqrt{1 + m^2 + n^2}\sqrt{1 + m'^2 + n'^2}}.$$

31. The angle between two lines may also be expressed in terms of the angles which each makes with the co-ordinate axes.

Let OA make with the axes of x, y, z the angles α, β, γ; and OB make with them the angles α', β', γ'; then, proceeding as in the last Art., we may shew that

$$\cos\theta = xx' + yy' + zz'.$$

But $x = \cos\alpha$, $y = \cos\beta$, $z = \cos\gamma$;

$x' = \cos\alpha'$, $y' = \cos\beta'$, $z' = \cos\gamma'$;

$$\therefore \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

32. Hence the condition in order that two lines may be perpendicular to one another is $\cos\theta = 0$, or

$$mm' + nn' + 1 = 0;$$

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at which we may arrive immediately by observing that the equation to a plane through the origin perpendicular to the first line is $z + mx + ny = 0$, which must necessarily contain a line through the same point parallel to the second whose equations are $x = m'z$, $y = n'z$;

$$\therefore z + mm'z + nn'z = 0, \quad \text{or } mm' + nn' + 1 = 0.$$

This condition expressed in another form, is

$$\cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' = 0.$$

Also, in order that the lines may be parallel to one another, we must have

$$\sin\theta = 0, \quad \text{or } (m - m')^2 + (n - n')^2 + (mn' - m'n)^2 = 0,$$

which can only be satisfied by $m = m'$, $n = n'$; this agrees with Cor. Art. 21.

33. To find the angle of inclination of two planes whose equations are given.

Let $z = Ax + By + c$, $z = A'x + B'y + c'$, be the equations to the two planes. Then the equations to two lines respectively perpendicular to them through the origin are (Art. 25.)

$$x = -Az, \quad y = -Bz; \quad x = -A'z, \quad y = -B'z;$$

and (Art. 27.) the angle of inclination, θ, of the two planes is equal to the angle between these two perpendiculars; therefore (Art. 30.)

$$\cos\theta = \frac{1 + AA' + BB'}{\pm\sqrt{1 + A^2 + B^2}\sqrt{1 + A'^2 + B'^2}};$$

the double sign belonging to the acute and obtuse angles between the two indefinite planes.

34. The angle of inclination of two planes may also be expressed in terms of the angles at which each is inclined to the co-ordinate planes.

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For if α, β, γ; α', β', γ'; denote the angles at which the planes are inclined to the co-ordinate planes of yz, zx, xy, these are also the angles which the perpendiculars on them from the origin respectively make with the axes of x, y, z; therefore (Art. 31.)

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

35. Hence the condition that two planes may be perpendicular to one another (since in that case $\cos\theta = 0$) may be expressed in the two following ways:

$$+\left\{\begin{array}{c} 1 + AA' + BB' = 0, \\ \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0. \end{array}\right.$$

36. To find the angle between a straight line and a plane, of which the equations are given.

The angle, θ, here meant, is the angle between the line and its projection upon the plane, and is therefore equal to the complement of the angle made by the line with a perpendicular to the plane.

Let $z = Ax + By + c$ be the equation to the plane; then the equations to a line through the origin perpendicular to it are

$$x = -Az, \quad y = -Bz.$$

Let $x = mz + a$, $y = nz + b$, be the equations to the given line; then since $\frac{\pi}{2} - \theta$ is the angle between it and the perpendicular to the plane,

$$\cos\left(\frac{\pi}{2} - \theta\right) \quad \text{or} \quad \sin\theta = \frac{1 - Am - Bn}{\sqrt{1 + m^2 + n^2}\sqrt{1 + A^2 + B^2}}.$$

37. To find the distance of a point from a plane.

Let x', y', z' be the co-ordinates of the given point P (fig. 9.), $z = Ax + By + c$ the equation to the given plane ABC.

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Through P draw a plane GKH perpendicular to the trace AB; then this plane is perpendicular both to the given plane ABC, and to AOB, and contains the ordinate PN. Let GH be the intersection of the two planes meeting PN in Q; draw PR perpendicular to GH. Then PR is perpendicular to the plane ABC; and $PR = PQ \sin PQR = PQ \cos GHK$.

$$\text{But } PQ = PN - QN = z' - (Ax' + By' + c),$$

since Q is a point in the plane ABC, for which $x = x'$, $y = y'$;

$$\text{and } \cos GHK = \frac{1}{\sqrt{1 + A^2 + B^2}} \quad \text{(Art. 29)};$$

$$\therefore PR, \text{ the required distance, } = \frac{z' - Ax' - By' - c}{\pm\sqrt{1 + A^2 + B^2}};$$

the radical being taken with that sign which makes the whole expression positive; for whenever distances are not measured parallel to a fixed line, the question can only relate to their absolute values.

Cor. Making $x' = 0$, $y' = 0$, $z' = 0$, we have the distance of the plane from the origin

$$= \frac{-c}{\pm\sqrt{1 + A^2 + B^2}},$$

where the radical must have the same sign as $-c$.

38. To find the distance of a point from a line.

Let P (fig. 10.) be the point, whose co-ordinates are x', y', z'; AB the line whose equations are $x = mz + a$, $y = nz + b$, so that a, b, are the co-ordinates of its trace A; PB a perpendicular from P on AB. Then

$$z - z' = -m(x - x') - n(y - y'),$$

$$\text{or } z = -mx - ny + z' + mx' + ny'$$

is the equation to a plane through P perpendicular to AB; therefore, by the preceding Art., since AB is drawn from a point $(a, b, 0)$ perpendicular to this plane,

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$$AB = \frac{ma + nb - (z' + mx' + ny')}{\sqrt{1 + m^2 + n^2}},$$

and $AP = \sqrt{(x' - a)^2 + (y' - b)^2 + z'^2}$;

$$\therefore BP^2 = AP^2 - AB^2$$

$$= (x' - a)^2 + (y' - b)^2 + z'^2 - \frac{\{z' + m(x' - a) + n(y' - b)\}^2}{1 + m^2 + n^2},$$

or, by reduction,

$$BP^2 = \frac{(x' - mz' - a)^2 + (y' - nz' - b)^2 + \{n(x' - a) - m(y' - b)\}^2}{1 + m^2 + n^2}.$$

Cor. The distance may also be obtained in terms of the angles which the line makes with the axes. Let α, β, γ; α', β', γ', be the angles which AB, AP, make respectively with the axes of x, y, z;

then $\cos PAB = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$

$$= \frac{(x' - a)\cos\alpha + (y' - b)\cos\beta + z' \cos\gamma}{AP} \quad \text{(Art. 6, Cor.)};$$

note. ✗

$$\therefore BP^2 = AP^2 - (AP \cos PAB)^2,$$

$$= (x' - a)^2 + (y' - b)^2 + z'^2$$

$$- \{(x' - a)\cos\alpha + (y' - b)\cos\beta + z' \cos\gamma\}^2.$$

39. As two straight lines in space, although not parallel, may never intersect, it is a Problem which sometimes arises to determine the shortest distance between them; it may be solved by means of the following proposition.

✗ 40. The shortest line which can join any two points in two straight lines in space, is the line which meets both of them at right angles.

Take one of the lines OC (fig. 11.) for the axis of x, and let the other AB meet the planes of zx, xy, in A and B,

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in which planes draw AO, Oy perpendicular to OC, and let these be the axes of x and y. Draw By parallel to Ox, join Ay and draw OM perpendicular to it; draw MN parallel to By, make Ox equal to MN and join Nx. Then MN being parallel to Ox is perpendicular to MO; therefore OM is perpendicular to each of the lines MN, Ay, and therefore to the plane AyB; also MN being equal and parallel to Ox, the figure NO is a rectangle, and Nx is equal and parallel to MO; therefore Nx is perpendicular to Ox, and also to the plane AyB and therefore to AB; hence Nx meets each of the given lines at right angles. It is also their shortest distance; for take any point K in AB, draw KL parallel to By and join OL; then the distance of K from Ox is equal to OL, because a perpendicular from K on Ox would be parallel and equal to OL, $\angle LOx$ being a right angle, and OL is manifestly greater than OM.

Cor. Any plane passing through the shortest distance will be perpendicular to the plane to which both the given lines are parallel; for since Nx is perpendicular to the plane AyB, any plane through Nx is perpendicular to AyB; and AyB, since it contains one and is parallel to the other, is itself parallel to every plane to which both the lines are parallel. Also if through OC we conceive a plane to be drawn parallel to AyB, it will be parallel to AB; and $MO = Nx$ will be its perpendicular distance from the plane AyB; *i. e.* the shortest distance between two lines not in the same plane is the perpendicular distance of two planes, each of which is drawn through one of the lines parallel to the other.

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41. We shall conclude this section with two or three Problems, which will furnish occasion to apply several of the results already obtained. We shall first complete the determination of the shortest distance between two lines, by finding its position and magnitude, when the equations to the lines are given.

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PROB. 1. To find the equations to a straight line which cuts perpendicularly each of two straight lines not in the same plane whose equations are given.

Let $\left.\begin{matrix} x = mz + a \\ y = nz + b \end{matrix}\right\}$, $\left.\begin{matrix} x = m'z + a' \\ y = n'z + b' \end{matrix}\right\}$ be the equation to the given lines; then if $z = Ax + By + c$ be the equation to a plane parallel to them both, $Am + Bn = 1$, $Am' + Bn' = 1$, (Art. 23.)

$$\therefore A = \frac{n' - n}{mn' - m'n}, \quad B = \frac{m' - m}{m'n - mn'}.$$

Let $z = A'x + B'y + c'$ be the equation to a plane which contains the first given line, and also the required line, and which therefore (Art. 40, Cor.) cuts the plane parallel to both the given lines at right angles; therefore (Art. 23. and 35.)

$$AA' + BB' + 1 = 0, \quad A'm + B'n - 1 = 0, \quad A'a + B'b + c' = 0;$$

$$\text{hence } c' = -A'a - B'b,$$

and $A'(An - Bm) = -(B + n)$, $B'(Bm - An) = -(A + m)$;

$$\therefore z = A'(x - a) + B'(y - b),$$

$$\text{or } z(An - Bm) = -(x - a)(B + n) + (y - b)(A + m),$$

or, substituting the values of A and B,

$$z\{m(m' - m) + n(n' - n)\} = (x - a)\{m' - m + n(m'n - mn')\}$$
$$+ (y - b)\{n' - n + m(mn' - m'n)\},$$

is the equation to the plane containing the first given line and the required line.

Similarly, the equation to the plane containing the second given line and the required line, by interchanging m, n, a, b, and m', n', a', b', is

$$z\{m'(m - m') + n'(n - n')\} = (x - a')\{m - m' + n'(mn' - m'n)\}$$
$$+ (y - b')\{n - n' + m'(m'n - mn')\}.$$

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These two equations determine the position of the line required, since they are the equations to two planes each of which contains it; the equations to its projections may be found by Art. 20.

PROB. 2. To find the length of the shortest distance between two straight lines whose equations are given.

If a plane be drawn through each of the lines parallel to the other, the perpendicular distance of these planes, which will of course be parallel to one another, will be the shortest distance of the lines (Art. 40. Cor.).

Let $\left.\begin{matrix} x = mz + a \\ y = nz + b \end{matrix}\right\}$, $\left.\begin{matrix} x = m'z + a' \\ y = n'z + b' \end{matrix}\right\}$ be the equations to the lines, $z = Ax + By + c$, the equation to the plane which is parallel to the second, and contains the first,

$$\therefore\ Am' + Bn' = 1,\quad Am + Bn = 1,\quad Aa + Bb + c = 0,$$

which give for A, B, c, the values

$$A = \frac{n' - n}{mn' - m'n},\quad B = -\frac{m' - m}{mn' - m'n},\quad c = -Aa - Bb.$$

Similarly, the equation to the plane which is parallel to the first, and contains the second, is

$$z = Ax + By + c',$$

where A, B have the above values, and

$$c' = -Aa' - Bb';$$

and the lengths of the perpendiculars dropped upon these planes from the origin are respectively (Art. 37. Cor.)

$$\frac{-c}{\sqrt{1 + A^2 + B^2}},\quad \frac{-c'}{\sqrt{1 + A^2 + B^2}},$$

the difference of which

$$= \frac{-(c' - c)}{\sqrt{1 + A^2 + B^2}} = \frac{A(a' - a) + B(b' - b)}{\sqrt{1 + A^2 + B^2}};$$

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therefore, substituting for A and B their values, the shortest distance between the two lines

$$= \frac{(n'-n)(a'-a) - (m'-m)(b'-b)}{\sqrt{(m'-m)^2 + (n'-n)^2 + (mn'-m'n)^2}}.$$

Cor. When c in the equation to the first plane, and c' the corresponding quantity for the second plane, have different signs, the origin is situated between the two planes; and therefore the distance of the planes will be found by taking the sum of the perpendiculars. When the shortest distance vanishes,

$$(n'-n)(a'-a) = (m'-m)(b'-b),$$

which is the condition in order that the lines may intersect, found in Art. 22.

Prob. 3. Having given the lengths, the least distance, and the inclination of two opposite edges of a tetrahedron, to find its volume.

Let A (fig. 11.) be the vertex, and BOC the base of any tetrahedron; complete the parallelogram Cy, and join Ay; let Nx cut each of the opposite edges AB, OC, at right angles; then OC is parallel, and Nx is perpendicular to the plane ABy. Hence, since they are on the same base, and their vertices are in a line parallel to the plane of base,

$$\text{tetrahedron } AyBO = \text{tetrahedron } AyBx$$

$$= \tfrac{1}{2} AB \,.\, By \,.\, \sin ABy \,.\, \tfrac{1}{3} Nx;$$

$$\text{but tetrahedron } AyBO = \text{tetrahedron } AOBC,$$

since they have a common vertex, and equal bases in the same plane;

$$\therefore \text{ tetrahedron } AOBC = \tfrac{1}{6} AB \,.\, OC \,.\, Nx \,.\, \sin\theta,$$

where θ denotes the angle between AB and OC.

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SECTION II.

ON SURFACES OF THE SECOND ORDER.

42. THE foregoing section comprehends all the more useful results relative to lines and planes referred to rectangular co-ordinates. We shall now proceed to investigate the equations and forms of surfaces of the second order, which, after the plane, are next in simplicity; reserving for the subject of the following section (after the reader has become familiar with the mode of representing surfaces by equations, and thence deducing their forms) one of the principal means of modifying and simplifying those equations, viz. transformation of co-ordinates; we shall then have occasion to introduce the projections of lines and plane surfaces; and the more useful results, relative to the line and plane referred to oblique co-ordinates, may be then more conveniently given than at present. It will be necessary however to make some preliminary observations on the geometrical meaning of equations containing only one or two of the variables x, y, z, when we embrace in our enquiries the three dimensions of space.

43. It was remarked (Art. 10.) that the equation $x = a$, when three dimensions of space are regarded, represents an indefinite plane parallel to yz; similarly, the equation $y^2 + py + q = 0$, which gives for y two constant values $y = a \pm b$, represents two planes parallel to zx; and in general, every equation containing only one of the variables, represents a system of planes parallel to the two axes whose co-ordinates it does not involve.

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Again, the equation $f(x, y) = 0$, which, in Plane Geometry, belongs to a curve CC' (fig. 12.) in the plane of xy, in Geometry of Three Dimensions, represents a cylindrical surface formed by drawing lines through all the points of CC' parallel to Oz; for any point P in this surface, whatever be the value of PC or z, will have the same x and y as its projection C; and therefore the co-ordinates of every point in the cylindrical surface will satisfy the relation $f(x, y) = 0$, which does not contain z; and any point whose projection is not in CC' cannot by its co-ordinates satisfy the relation $f(x, y) = 0$. The same observations apply to equations of the form $f(x, z) = 0$, $f(y, z) = 0$. Hence, an equation containing only two of the variables represents a cylindrical surface parallel to that axis whose co-ordinate it does not involve; and of which the trace upon the plane containing the other two axes, is in Plane Geometry given by the same equation.

44. The locus of an equation $f(x, y, z) = 0$, containing all the variables, is a surface; that is, if all the points be taken whose co-ordinates satisfy it, they will not comprehend all the points of a solid figure, but only those situated in a surface. In the equation $f(x, y, z) = 0$ we may assume any values for two of the co-ordinates; and, deducing the value of the third from it, we have the co-ordinates of the corresponding point in the locus; and similarly, we may determine as many points in the locus as we please. Instead of this, however, let us assume the value of only one co-ordinate $x = a$; then (Art. 43.) $f(y, z, a) = 0$, represents a cylindrical surface parallel to the axis of x; but since we must only take the points in this surface which satisfy the condition $x = a$, it follows that by this assumption we obtain a *curve*, namely, the section of the cylinder made by the plane $x = a$ parallel to yz. Let this be PQ (fig. 13.), where $ON = a$; then of all the points in the indefinite plane PNQ it is only those in the curve PQ which are determined by the equation $f(y, z, a) = 0$; similarly, if we take $x = ON' = a'$, we shall have $f(y, z, a') = 0$, which equations determine the curve $P'Q'$; and, proceeding in this manner,

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we may obtain an infinite number of curves all situated in planes parallel to yz, and succeeding one another at intervals as small as we please, the assemblage of which will form a surface which is the locus of the equation $f(x, y, z) = 0$.

45. We conclude, therefore, that every equation, whether containing one, two, or three of the variables, represents a surface; if however the equation can be satisfied by no system of real values of the co-ordinates, the surface will be altogether imaginary; or, if it can be satisfied only by breaking it up into two or three others, the surface is reduced to a limited number of lines or points. In other cases the equation will enable us to determine the figure and properties of the surface which is its locus, as will be seen. In discovering the figure from the equation, the traces on the co-ordinate planes, determined by putting x, y, z separately $= 0$ in the equation to the surface, and the sections made by planes parallel to the co-ordinate planes, determined by putting the co-ordinates each separately equal to a constant, are the principal means. Thus (fig. 13.) AB, BC, CA are the three traces of the surface; and PQ is a section made parallel to yz, the equation to which is $f(y, z, a) = 0$, where $ON = a$, and QN, NP are to be regarded as the axes to which the curve PQ is referred. As in Geometry of Two Dimensions, a curve may have several branches, and an ordinate several values corresponding to the same value of the abscissa; so a surface may have several sheets, and points in different sheets may have the same projection on the co-ordinate planes. As many real values as z has for given values of x and y, so many sheets will the surface have; if the value of z is imaginary, there is no point in the surface corresponding to those values of x and y.

46. Surfaces, in the same manner as lines, are divided into *orders*, according to the degree of their equations; the degree being determined by the sum of the indices of the three variables in that term of the equation where it is

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greatest. The plane is the surface of the first order, being the locus of the equation of the first degree between three variables; the sphere, the common cone and cylinder, the ellipsoid, &c., are surfaces of the second order, because their equations are of the second degree. We shall at present obtain the equation to each variety of the surfaces of the second order from some known properties of them, and determine their figures; reserving the discussion of the general equation of the second order, and the closer consideration of the properties of the surfaces which it represents, to a more advanced part of the work. This arrangement, which corresponds to that usually followed with regard to curves of the second order, will gradually introduce the student to the more difficult parts of the subject, and put him at once in possession of the more important results.

47. To find the equation to the surface of a sphere.

The characteristic property of this surface, is, that every point in it is at the same distance from the center.

Let x, y, z be the co-ordinates of any point in the surface, a, b, c the co-ordinates of its center, and r its radius; then

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2,$$

is the relation which the co-ordinates of every point must satisfy, and therefore the equation to the surface.

Cor. This equation will assume different forms according to the position of the origin; thus if the origin be in the center, $a=b=c=0$, and the equation becomes

$$x^2+y^2+z^2=r^2;$$

if the origin be in the surface of the sphere,

$$a^2+b^2+c^2=r^2,$$

and the equation becomes

$$x^2+y^2+z^2=2(ax+by+cz).$$

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48. To find the equation to the surface of an oblique cylinder, the base being circular.

This surface is generated by an indefinite line which is carried round the perimeter of a given circle, always remaining parallel to a given straight line.

Let $x = mz$, $y = nz$, be the equations to a line OR (fig. 14.) through the origin, to which the generating line is always parallel; and r the radius of the circle OPA described in the plane of xy with its diameter coinciding with the axis of x, along which the generating line moves. Let PQ be the generating line in any position, and x, y, z the co-ordinates of any point M in it, and therefore in the cylindrical surface. Then the equations to PQ are

$$X - x = m(Z - z), \quad Y - y = n(Z - z), \quad \text{(Cor. Art. 21.)};$$

therefore, making $Z = 0$, we have the co-ordinates of P, or

$$X' = ON = x - mz, \quad Y' = NP = y - nz.$$

But, P being a point in the circular base,

$$Y'^2 = 2rX' - X'^2;$$

therefore, substituting for Y' and X' their values,

$$(y - nz)^2 = 2r(x - mz) - (x - mz)^2,$$

a relation among the co-ordinates of any point in the surface, and therefore its equation.

Cor. We may suppose the curve OP, instead of a circle, to be any curve of the second order determined by the equation $Y'^2 = lX' - (1 - e^2)X'^2$; we shall then obtain the equation to any cylindrical surface of the second order, viz.

$$(y - nz)^2 = l(x - mz) - (1 - e^2)(x - mz)^2.$$

And it is easily seen that whatever the curve OP be, if its equation be $Y' = f(X')$, the equation to the cylindrical surface generated by a straight line carried along it, and of which it is called the directrix, will be $(y - nz) = f(x - mz)$.

49. To find the equation to the surface of an oblique cone, the base being circular. *See Part 7. p. 91*

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This surface is generated by an indefinite line which is carried round the perimeter of a given circle, always passing through a fixed point.

Let a, b, c be the co-ordinates OR, RQ, QV (fig. 15.) of the vertex V, and r the radius of the circle OPA described in the plane of xy with its diameter coinciding with the axis of x, along which the generating line moves. Let PV be the generating line in any position, and x, y, z the co-ordinates of any point M in it, and therefore in the conical surface; then the equations to VP are (Art. 21.)

$$X - a = \frac{x-a}{z-c}(Z-c), \quad Y - b = \frac{y-b}{z-c}(Z-c);$$

therefore making $Z = 0$, we have the co-ordinates of P, or

$$X' = ON = a - \frac{x-a}{z-c}c, \quad Y' = NP = b - \frac{y-b}{z-c}c.$$

But, P being a point in the circular base,

$$Y'^2 = 2rX' - X'^2;$$

therefore, substituting and reducing,

$$(bz - cy)^2 = 2r(z-c)(az - cx) - (az - cx)^2,$$

or, $(bz-cy)^2 = (cx-az)(2cr - 2rz + az - cx)$.

the equation to the conical surface.

Cor. We may suppose the curve OP, instead of a circle, to be any curve of the second order, determined by the equation $Y'^2 = lX' - (1-e^2)X'^2$; we shall then obtain the equation to any conical surface of the second order, viz.

$$(bz - cy)^2 = l(z-c)(az - cx) - (1-e^2)(az - cx)^2.$$

And it is manifest that whatever the curve OP be, if its equation be $Y' = f(X')$, the equation to the conical surface of which it is the directrix will be

$$\frac{bz - cy}{z-c} = f\left(\frac{az - cx}{z-c}\right).$$

Cone of the second degree

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50. To find the equation to the surface of a spheroid.

This surface is generated by the revolution of an ellipse about one of its axes.

Let CA (fig. 16.) be a quadrant of an ellipse which, by revolving about its axis OC coinciding with the axis of z, generates a spheroid. Let $A'C$ be any position of the generating curve, and $ON = x$, $NM = y$, $MP' = z$ the co-ordinates of any point P' in it; $AO = a$, $CO = c$; then from the right-angled triangle ONM,

$$OM^2 = x^2 + y^2.$$

But, since P' is a point in an ellipse with $\frac{1}{2}$ axes a and c,

$$\frac{OM^2}{a^2} + \frac{z^2}{c^2} = 1;$$

Spheroid $\left\{ \right.$ $\therefore \frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1,$

the equation to the surface; which is called an *oblate* or *prolate* spheroid, according as the axis $2c$, about which the ellipse revolves, is the less or greater of its axes.

Cor. Similarly, if the hyperbola AQ (fig. 18.) revolve about its conjugate axis OC coinciding with the axis of z, the equation to the surface of the hyperboloid generated will be

Hyperboloid of one sheet. $\left\{ \right.$ $\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1;$

and if it revolve about its transverse axis OA (fig. 19.) coinciding with the axis of x, the equation to the surface generated will be

Hyperboloid of two sheets. $\left\{ \right.$ $\frac{x^2}{a^2} - \frac{y^2 + z^2}{c^2} = 1.$

51. To find the equation to the surface of a paraboloid.

Let OQ (fig. 20.) be a parabola which by revolving about its axis, coincident with the axis of x, generates a paraboloid; OP any position of the generating curve,

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$$ON = x, \quad NM = y, \quad MP = z,$$

the co-ordinates of any point P in it; then

$$PN^2 = l \,.\, ON = lx,$$

l being the latus rectum of OP; but from the right-angled triangle PMN,

$$PN^2 = y^2 + z^2,$$

Paraboloid $+\{ \quad \therefore y^2 + z^2 = lx,$

the equation to the surface.

52. When, as in the preceding instances, a plane curve revolves about an axis coincident with one of the co-ordinate axes, the equation to the surface generated may be readily obtained, whatever be the generating curve. For let OC (fig. 16.) the axis of x be the axis of revolution, CPA the generating curve, and $PR = f(OR)$ its equation; and let it revolve about OC into the position $CP'A'$; $ON = x$, $NM = y$, $MP' = z$, the co-ordinates of any point P'.

Then by the equation to the curve, $MP' = f(OM)$,

$$\text{or} \quad z = f(\sqrt{x^2 + y^2}),$$

the equation to the surface.

53. The surfaces which we have hitherto considered, are the simplest cases of surfaces of the second order; their equations are all contained in the general equation of the second order between three variables, which is

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0.$$

It will be shewn hereafter that this equation, by giving a proper position and directions to the origin and axes of the co-ordinates, can always be reduced to one of the forms

$$Ax^2 + By^2 + Cz^2 = + D,$$

$$By^2 + Cz^2 = 2A'x,$$

which represent two distinct families of surfaces; the former those which have a center, the latter those which have not

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a center; meaning by center, a point such that all chords of the surface drawn through it are bisected in it.

54. The origin is the center, and the co-ordinate planes are the principal planes, of the surface represented by the equation

$$Ax^2 + By^2 + Cz^2 = +D.$$

For let P (fig. 22.) be a point in the surface whose co-ordinates are $x = h$, $y = k$, $z = l$; then the equation is satisfied by these values, and therefore it is also satisfied when $-h$, $-k$, $-l$ are written for x, y, and z. But if we produce PO to P', and make $OP' = OP$, the co-ordinates of P' are evidently equal to $-h$, $-k$, $-l$; therefore P' is a point in the surface, and PP' is a chord, and it is bisected in O; *i. e.* every chord is bisected in O, therefore O is the center of the surface. Also the surface is situated symmetrically with respect to the co-ordinate planes. For if in the equation we make $x = ON = h$, and $y = NM = k$, it will give for z two equal values with contrary signs PM, QM; so that for every point of the surface, situated above the plane of xy, there will be a corresponding point situated at an equal distance below it. Similarly, the other co-ordinate planes may be shewn to bisect all their ordinates at right angles. Planes which have this property are called the *principal planes* of the surface, and their intersections (which in this case coincide with the axes of the co-ordinates, and with respect to each of which also the surface is symmetrically situated) the *axes* of the surface. Also every chord passing through the center is called a *diameter* of the surface.

55. In the equation to surfaces that have a center

$$Ax^2 + By^2 + Cz^2 = +D,$$

having taken care to make the second member positive, since the coefficients of the variables cannot be all negative together, we can only have three varieties of form, (1) all the coefficients positive, (2) one negative, (3) two negative; so that the equation may assume the three following forms:

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Ellipsoid (1.) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$

Hyperboloid (2.) of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$ last eq = 0 is becomes a cone whose base is an ellipse.

Hyperboloid of two sheets (3.) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

The surfaces represented by them are called respectively the ellipsoid, (1.) the hyperboloid of one sheet, (2.) and the hyperboloid of two sheets. (3.)

56. In the family of surfaces, represented by the equation

$$By^2 + Cz^2 = 2A'x,$$

the origin is not the center, since the equation does not remain unchanged when x, y, z are changed into $-x$, $-y$, $-z$; and it will be seen hereafter that no other point can be its center. Also since only even powers of y and z enter the equation, the planes of zx and xy are principal planes, but the plane of yz not a principal plane; consequently the surface has only one axis, coinciding with the axis of x. In the equation, if B and A' are not both positive, let the coefficient of y^2 be made positive; and then, if necessary, change the sign of x in order to make the second member positive, for the change of x into $-x$ will merely alter the position of the surface; the equation will then offer two varieties according as C is positive or negative, under the forms

see (72.) page 51

$$\frac{y^2}{l} + \frac{z^2}{l'} = x, \quad \text{Elliptic paraboloid}$$

$$\frac{y^2}{l} - \frac{z^2}{l'} = x; \quad \text{Hyperbolic paraboloid}$$

and the surfaces represented by them are called respectively the elliptic and hyperbolic paraboloids.

57. To find the equation to the surface of an ellipsoid.

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This surface is generated by a variable ellipse which moves parallel to itself with its axes in two fixed planes, and vertices in two *ellipses* in those planes having a common axis coincident with the intersection of the planes.

Let BC, CA (fig. 17.) be quadrants of the given ellipses traced in the planes yz, zx; $OC = c$ their common $\frac{1}{2}$ axis coinciding with the axis of z, $OA = a$, $OB = b$, the other $\frac{1}{2}$ axes; QPR a quadrant of the generating ellipse in any position, having its plane parallel to xy, its center in OC, and two of its vertices in the ellipses AC, BC, so that the ordinates QN, RN are its $\frac{1}{2}$ axes; also let $ON = z$, $NM = x$, $MP = y$ be the co-ordinates of any point P in it; then

$$\left(\frac{NQ}{a}\right)^2 = \left(\frac{NR}{b}\right)^2, \text{ because each} = 1 - \left(\frac{ON}{c}\right)^2;$$

$$\therefore y^2 = \left(\frac{NR}{NQ}\right)^2 (NQ^2 - x^2) = NR^2 - \frac{b^2}{a^2}x^2 = b^2\left(1 - \frac{z^2}{c^2}\right) - \frac{b^2}{a^2}x^2,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation to the surface.

58. To determine the form of the ellipsoid from its equation.

Since in the equation, x can only receive values between a and $-a$, y between b and $-b$, and z between c and $-c$, the surface is limited in all directions. If we put $z = 0$, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for the equation to the trace on xy, which is therefore an ellipse AB; also, from the mode of generation, all sections parallel to xy are ellipses, and since their axes are in the ratio of a to b, they are all similar to AB.

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If we make $x = \pm h$, we have

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{h^2}{a^2}$$

for the equation to any section parallel to yz, which is an ellipse similar to the trace BC, since its axes are in the ratio of b to c whatever be the value of h, and which becomes imaginary when $h > a$. In the same manner it may be shewn, that all sections parallel to zx are ellipses, similar to the trace AC. The ellipses, of which AB, BC, CA are quadrants, in which the surface is intersected by the co-ordinate planes, which are its principal planes, are called the *principal sections* of the surface; and the parts of the co-ordinate axes intercepted within the surface, are called its axes. Hence a, b, c represent the $\frac{1}{2}$ axes of the ellipsoid, and also the $\frac{1}{2}$ axes of its principal sections. The extremities of the axes such as A, B, C are called the *vertices* of the ellipsoid, of these it has six, one at the extremities of each axis.

The whole surface consists of eight portions precisely similar and equal to that represented in the figure.

Cor. If $a = b$, the equation becomes that to a spheroid generated by revolution about Oz; similarly, if any other two of the semiaxes become equal, the ellipsoid becomes a spheroid generated by revolution about the remaining axis.

59. To find the equation to the surface of a hyperboloid of one sheet.

This surface is generated by a variable ellipse which moves parallel to itself, with its axes in two fixed planes, and vertices on two *hyperbolas* in those planes having a common *conjugate* axis coincident with the intersection of the planes.

Let AQ, BR (fig. 18.) be the given hyperbolas traced in the planes zx, yz; $OC = c$ their common $\frac{1}{2}$ conjugate axis coinciding with the axis of z, $OA = a$, $OB = b$, the

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$\frac{1}{2}$ transverse axes; QPR the generating ellipse in any position, having its plane parallel to xy, its center in OC, and its vertices in the hyperbolas AQ, BR, so that the ordinates NQ, NR, are its $\frac{1}{2}$ axes. Also let $ON = z$, $NM = x$, $MP = y$ be the co-ordinates of any point P in the generating ellipse; then

$$\left(\frac{NQ}{a}\right)^2 = \left(\frac{NR}{b}\right)^2; \text{ since each } = 1 + \left(\frac{ON}{c}\right)^2;$$

$$\therefore y^2 = \left(\frac{NR}{NQ}\right)^2 (NQ^2 - x^2) = NR^2 - \frac{b^2}{a^2}x^2 = b^2\left(1 + \frac{z^2}{c^2}\right) - \frac{b^2}{a^2}x^2,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the equation to the surface.

60. To determine the form of the hyperboloid of one sheet from its equation.

Since the equation will visibly admit values of x, y, z positive and negative however large, the surface is extended indefinitely on all sides of the origin. If we put $z = 0$, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for the equation to the trace on xy, which is the ellipse AB; and from the mode of generation all sections parallel to xy are similar ellipses, the dimensions of which increase indefinitely, the least being that of which AB is a quadrant, and which forms the interior limit of the surface. For the sections parallel to yz, putting $x = \pm h$, we have

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{h^2}{a^2},$$

which, as long as $h < a$, and therefore the second member positive, represents a hyperbola similar to the trace BR, with its vertices in the ellipse AB, and conjugate axis parallel to Oz, since the coefficient of z^2 is negative. When

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$h = a$, the equation represents two straight lines; and when $h > a$, making the second member positive, the equation is

$$\frac{x^2}{c^2} - \frac{y^2}{b^2} = \frac{h^2}{a^2} - 1,$$

which also represents a hyperbola, but in a new position, namely, with its vertices in AQ, and conjugate axis parallel to Oy. In the same manner, the sections parallel to xx may be shewn to be hyperbolas similar to the trace AQ. The principal sections of this surface are the ellipse AB, and the hyperbolas AQ, BR. The quantities a, b, which denote the distances from the origin at which the surface cuts the axes of x and y, are called the *real* semiaxes of the surface; the quantity c is called the *imaginary* semiaxis, because if we put $x = 0$, $y = 0$, to find when the surface cuts the axis of z, we find

$$z^2 = -c^2, \quad \text{or} \quad z = \pm c\sqrt{-1},$$

so that c is the coefficient of the expression for the imaginary $\frac{1}{2}$ axis of the surface. The extremities of the real axes are called the vertices of the surface; two of them are A, B, and the remaining two are at distances from O respectively equal to a and b, in AO and BO produced. The whole surface consists of eight portions, precisely similar and equal to that represented in the figure; and since it is continuous, that is, since we can pass from one point in it to any other point in it, without quitting the surface, it is called the hyperboloid of one sheet.

Cor. If $a = b$, all sections parallel to xy become circles, and the surface becomes a hyperboloid of revolution about the conjugate axis.

61. The hyperboloid of one sheet has an interior conical asymptote.

Putting the equation under the form

$$\frac{x^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(1 - \frac{a^2b^2}{a^2y^2 + b^2x^2}\right),$$

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we see that when x and y are very great, the quantity

$$\frac{a^2b^2}{a^2y^2+b^2x^2}$$

is very small; and therefore the relation among the co-ordinates of points very distant from the origin, is nearly expressed by

$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2};$$

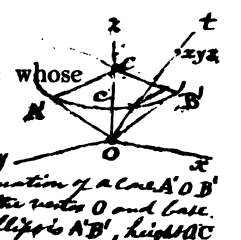
this then is the equation to a surface, whose distance from the surface of the hyperboloid, measured parallel to the axis of z, diminishes indefinitely as x and y increase; and, as its ordinate is always greater than the corresponding ordinate of the hyperboloid, it lies within the latter surface. This asymptotic surface is a right cone vertex O, and base an ellipse $A'B'$, whose center is in C, and $\frac{1}{2}$ axes equal and parallel to AO, BO; for if x, y, z be co-ordinates of any point in a generating line of this conical surface, its equations will be

$$X=\frac{x}{z}Z, \quad Y=\frac{y}{z}Z;$$

$\therefore$ the co-ordinates of the point where it meets $A'B'$ are, making $Z=c$,

$$X'=\frac{x}{z}c, \quad Y'=\frac{y}{z}c;$$

and as these are co-ordinates of a point in an ellipse whose $\frac{1}{2}$ axes are a and b,



$$\left\{\left(\frac{x}{z}\frac{c}{a}\right)^2+\left(\frac{y}{z}\frac{c}{b}\right)^2=1 \quad \text{or} \quad \frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}\right.$$

the equation of a cone $A'OB'$ with the vertex O and base the ellipse $A'B'$, height $=c$ and a and b the semi axes.

62. To find the equation to the surface of a hyperboloid of two sheets.

This surface is generated by a variable ellipse which moves parallel to itself, with its axes in two fixed planes, and vertices in two *hyperbolas* in those planes having a

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common *transverse* axis, coincident with the intersection of the planes.

Let AQ, AR (fig. 19.) be the given hyperbolas traced in the planes zx, xy; $OA = a$, their common $\frac{1}{2}$ transverse axis coinciding with the axis of x, $OB = b$, $OC = c$, the $\frac{1}{2}$ conjugate axes; QPR the generating ellipse in any position, having its plane parallel to yz, its center in Ox, and its vertices in AQ, AR, so that the ordinates QN, RN are its semiaxes. Let $ON = x$, $NM = y$, $MP = z$ be the co-ordinates of any point P in the ellipse,

$$\text{then } \left(\frac{QN}{c}\right)^2 = \left(\frac{RN}{b}\right)^2, \text{ since each} = \left(\frac{ON}{a}\right)^2 - 1;$$

$$\therefore\ z^2 = \left(\frac{QN}{RN}\right)^2 (RN^2 - y^2) = QN^2 - \frac{c^2}{b^2} y^2 = c^2 \left(\frac{x^2}{a^2} - 1\right) - \frac{c^2}{b^2} y^2;$$

$$\text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the equation to the surface.

63. To determine the form of the hyperboloid of two sheets from its equation.

The equation shews that all values of x, between $+a$ and $-a$, are inadmissible, therefore no part of the surface can be situated between two planes parallel to yz through A and A'; but the equation can be satisfied by values of x, y, z indefinitely great, therefore there is no limit to the distance to which the surface may extend from O. If we put $x = 0$, we have

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1,$$

therefore the principal section by yz is imaginary; but all sections parallel to yz, and at a distance from it greater than a, are similar ellipses as appears from the mode of generation. For the sections parallel to xy, putting $z = \pm l$, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{l^2}{c^2}.$$

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which represents a hyperbola similar to the principal section AR with its vertices in AQ and the opposite branch, and conjugate axis parallel to Oy; and in the same way it may be shewn, that the sections parallel to zx are hyperbolas similar to the principal section AQ with vertices in AR and the opposite branch, and conjugate axes parallel to Oz. If we make at once $y = 0$ $z = 0$, to find where the surface cuts the axis of x, we have $x^2 = a^2$ or $x = \pm a$; $2a$ is the *real* axis of the surface, and its extremities the vertices of the surface of which there are but two; the quantities b, c are the imaginary semiaxes of the surface, because it does not cut either the axis of y or z. The whole surface consists of two sheets perfectly similar and equal, and indefinitely extended in opposite directions, but separated by an interval in which exists no point of the surface; it is therefore called the hyperboloid of two sheets; each sheet consists of four portions precisely similar and equal to that represented in the figure.

Cor. If $b = c$, or the two imaginary axes become equal, this surface becomes a hyperboloid of revolution about the transverse axis.

64. The hyperboloid of two sheets has an exterior conical asymptote.

Putting the equation under the form

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2} + 1 = \left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(1 + \frac{b^2c^2}{c^2y^2 + b^2z^2}\right),$$

we have, for points very distant from the origin, the relation among the co-ordinates nearly expressed by

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2};$$

this then is the equation to a surface whose distance from the hyperboloid, measured parallel to the axis of x, diminishes indefinitely as y and z increase; and, as its ordinate x is always less than the corresponding ordinate of the hyperbo-

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loid, the latter surface lies within it. It may be shewn, as in Art. 61, that the asymptotic surface is a right cone vertex O, and base an ellipse $B'C'$ whose center is A and semiaxes parallel and equal to OB, OC. In this case, as in the former, we observe that the equation to the conical asymptote is obtained by omitting the constant term in the equation to the surface.

65. Since from the equation to the ellipsoid, the equation to the hyperboloid of one sheet results by changing c into $c\sqrt{-1}$; and the equation to the hyperboloid of two sheets, by changing b into $b\sqrt{-1}$ and c into $c\sqrt{-1}$; if any result in terms of its axes be obtained for the ellipsoid, the corresponding results for the hyperboloids may be deduced by writing

$$c\sqrt{-1} \text{ for } c, \text{ or } b\sqrt{-1},\ c\sqrt{-1}, \text{ for } b \text{ and } c.$$

66. To find the equation to the surface of the elliptic paraboloid. *Elliptic paraboloid.*

This surface is generated by a parabola which moves with its plane perpendicular to a fixed plane, and axis in that plane, and parallel to the axis of another parabola along which its vertex moves; the concavities of the parabolas being turned towards *the same* parts.

Let OR (fig. 20.) be a parabola traced in the plane of xy, vertex at the origin, and axis coinciding with the axis of x, l its latus rectum; RP the generating parabola in any position with its plane parallel to zx, vertex in OR, and axis parallel to Ox, and let l' denote its latus rectum, and $ON = x$, $NM = y$, $MP = z$ be the co-ordinates of any point P in it; also draw RM' parallel to Oy. Then

$$z^2 = l'.RM = l'(ON - OM') = l'\left(x - \frac{y^2}{l}\right),$$

$$\therefore\ x = \frac{y^2}{l} + \frac{z^2}{l'},$$

the equation to the surface.

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67. To determine the form of the elliptic paraboloid from its equation.

Since only positive values of x are admissible, no part of the surface is situated to the left of the plane of yz; also since the equation can be satisfied by positive values of x, and by positive and negative values of y and z however large, the surface is extended indefinitely towards the positive direction of x. If we make $y = 0$, $z^2 = l'x$ is the equation to the principal section OQ, and from the mode of generation all sections parallel to zx are parabolas equal to OQ with vertices in OR; similarly, all sections parallel to xy are parabolas equal to the other principal section OR, and having their vertices in OQ. If we make $x = 0$, we find $l'y^2 + lz^2 = 0$, $\therefore y = 0$, $z = 0$, or the trace on yz is a point namely the origin. If we make $x = h$, we have

$$\frac{y^2}{lh} + \frac{z^2}{l'h} = 1$$

for the equation to a section parallel to yz, which represents an ellipse whose semiaxes are the ordinates QN, NT which are to one another in a constant ratio $\sqrt{l'}$ to $\sqrt{l}$; therefore all sections perpendicular to the axis of the surface, are similar ellipses, and hence the name of the surface. It has only one vertex O and one indefinite axis Ox, and consists of one sheet; and is reduced to a paraboloid of revolution when $l = l'$.

68. To find the equation to the surface of the hyperbolic paraboloid.

Hyperbolic paraboloid.

This surface is generated by a parabola which moves with its plane perpendicular to a fixed plane, and axis in that plane, and parallel to the axis of another parabola along which its vertex moves; the concavities of the parabolas being turned towards *opposite* parts.

Let OR (fig. 21.) be a parabola in the plane of xy, vertex at the origin, and axis coinciding with the axis of x;

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and l its latus rectum; RP the generating parabola in any position with its plane parallel to zx, vertex in OR, and axis parallel to Ox, and let l' denote its latus rectum, and $ON = x$, $NM = y$, $MP = z$ the co-ordinates of any point P in it; draw RM' parallel to Oy, then

$$z^2 = l' \cdot MR = l' (OM' - ON) = l' \left(\frac{y^2}{l} - x \right);$$

$$\therefore x = \frac{y^2}{l} - \frac{z^2}{l'},$$

the equation to the surface.

69. To determine the form of the hyperbolic paraboloid from its equation.

The surface cuts the co-ordinate axes only at the origin, and since the equation will visibly admit values of x, y, z positive and negative however large, the surface is extended indefinitely from the origin. If we make $y = 0$, we have $z^2 = -l'x$ for the equation to the trace on zx, which represents the parabola OQ, with its concavity turned towards the negative direction of x, since its equation shews that x must be taken negatively; and from the mode of generation, all sections parallel to zx are parabolas equal to OQ, with their vertices in OR. The other principal section of the surface in the plane of xy, is the parabola OR, forming the interior limit of the figure, and all sections parallel to xy are parabolas equal to OR, with their vertices in OQ. For the trace on yz, putting $x = 0$, we have $z\sqrt{l} = \pm y\sqrt{l'}$, which represents two straight lines passing through the origin; and for sections parallel to yz, making $x = h$, we have

$$\frac{y^2}{lh} - \frac{z^2}{l'h} = 1,$$

which represents a hyperbola with its vertices in OR, and conjugate axis parallel to Oz. If we make h negative, the equation to the section becomes

$$\frac{z^2}{l'h} - \frac{y^2}{lh} = 1,$$

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which also represents a hyperbola, but in a new position, viz. with its vertices in OQ, and conjugate axis parallel to Oy. The surface consists of one sheet, and has only one vertex O, and one indefinite axis Ox; and all sections perpendicular to its axis are similar hyperbolas, whence its name. It does not become a surface of revolution when $l = l'$, nor in any other case, but the sections are eqh. hyperbolas.

70. The hyperbolic paraboloid has plane asymptotes.

Putting the equation under the form

$$z^2 = \frac{l'}{l}y^2 - l'x = \frac{l'}{l}y^2\left(1 - \frac{lx}{y^2}\right),$$

we have the relation, among the co-ordinates of points very distant from the origin, nearly expressed by

$$z^2 = \frac{l'}{l}y^2; \quad \therefore z = \pm y\sqrt{\frac{l'}{l}}$$

for Cos a =

is the equation to two planes through the origin perpendicular to the plane of yz, the distance between which and the paraboloid continually diminishes. These planes contain the asymptotes to all the hyperbolic sections parallel to yz, which, as we have seen, have their centers in the axis of x, and axes in the ratio of $\sqrt{l'}$ to $\sqrt{l}$.

71. The equation to the hyperbolic differs from that to the elliptic paraboloid, in having $-l'$ instead of l'; this relation will enable us to modify all results obtained for one surface, so as to be true for the other.

72. The elliptic and hyperbolic paraboloids are particular cases of the ellipsoid, and hyperboloid of one sheet respectively; viz. when the centers of these surfaces are removed to an infinite distance.

see (56.) page 40.

In the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1,$$

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write $x - a$ instead of x, then the resulting equation

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, \quad \text{or} \quad \frac{x^2}{a^2} - \frac{2x}{a} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0,$$

is reckoned from the vertex of the surface. Let p and p' denote the distances of the foci of the principal sections in xy and zx, from the vertex

$$\therefore b^2 = a^2 - (a-p)^2 = 2ap - p^2, \quad c^2 = 2ap' \mp p'^2;$$

hence, by substitution,

$$\frac{x^2}{a^2} - \frac{2x}{a} + \frac{y^2}{2ap - p^2} \pm \frac{z^2}{2ap' \mp p'^2} = 0,$$

$$\text{or } \frac{x^2}{a} - 2x + \frac{y^2}{2p - \frac{p^2}{a}} \pm \frac{z^2}{2p' \mp \frac{p'^2}{a}} = 0;$$

therefore making a infinite, *i. e.* supposing the center of the surface to remove to an infinite distance from the vertex, whilst the distances of the foci of the principal sections from their common vertex remain finite, we have

$$\frac{y^2}{2p} \pm \frac{z^2}{2p'} - 2x = 0,$$

which coincides with the equations to the paraboloidal surfaces.

Cor. Hence if any result be obtained for the ellipsoid or hyperboloid, it will be adapted to the paraboloids, by the modification above indicated; viz. by transferring the origin to the extremity of an axis, and making that axis infinite. Also both families may be represented by the equation

$$A x^2 + B y^2 + C z^2 = 2A'x,$$

the origin being at a vertex, and $A = 0$ when the surface has not a center.

73. Surfaces of the second order, admit of another division, viz. into those which can be generated by the motion of a straight line, and into those which cannot. This property which we have seen to belong to the cylinder and

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cone, we shall now shew to be possessed by the hyperboloid of one sheet and the hyperbolic paraboloid; it is easily foreseen that it cannot belong to the remaining surfaces of the second order, as their forms manifestly preclude their having a straight line applied to them throughout its indefinite length; this will also appear from our results.

74. The hyperboloid of one sheet can have an infinite number of straight lines entirely coinciding with its surface.

Let the equations to the surface and to a line be respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

$$x = mz + h, \quad y = nz + k;$$

if the line coincide with the surface in all its points, the equation

$$\frac{(mz+h)^2}{a^2} + \frac{(nz+k)^2}{b^2} - \frac{z^2}{c^2} = 1,$$

will be true for all values of z; therefore, equating to nothing the coefficients of z^2 and z, we have

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} - \frac{1}{c^2} = 0,$$

$$\frac{mh}{a^2} + \frac{nk}{b^2} = 0, \quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1.$$

Eliminate k between the two latter equations, and reduce by means of the former;

$$\therefore \frac{h^2}{a^2} + \left(\frac{mb}{na} \cdot \frac{h}{a}\right)^2 = 1, \quad \text{or} \quad \left(\frac{hb}{nac}\right)^2 = 1;$$

$$\therefore h = \pm \frac{nac}{b}, \quad \text{and} \quad k = \mp \frac{mbc}{a};$$

h & k are symmetric in the equations.

therefore the two lines whose equations are respectively

$$\left.\begin{aligned} x &= mz + \frac{nac}{b} \\ y &= nz - \frac{mbc}{a} \end{aligned}\right\} \ldots(1), \qquad \left.\begin{aligned} x &= mz - \frac{nac}{b} \\ y &= nz + \frac{mbc}{a} \end{aligned}\right\} \ldots(2),$$

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will coincide with the surface of the hyperboloid in all their points, m and n being any quantities which satisfy the equation

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{1}{c^2};$$

and if m be made to assume all values, and n be always taken $= \frac{b}{ac}\sqrt{a^2 - m^2c^2}$, we shall determine two infinite systems of lines represented by the equations (1) and (2), having the aforesaid property.

Cor. No two lines in the same system intersect; for if we take two lines in the first system represented by the equations

$$\left.\begin{aligned} x &= mz + \frac{nac}{b} \\ y &= nz - \frac{mbc}{a} \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= m'z + \frac{n'ac}{b} \\ y &= n'z - \frac{m'bc}{a} \end{aligned}\right\},$$

we shall find (Art. 22.) that they cannot intersect unless $(m - m')^2 = 0$, that is, unless they become identical. But any line in the first system intersects every line in the second system; for let the equations to two lines, one in each system, be

$$\left.\begin{aligned} x &= mz + \frac{nac}{b} \\ y &= nz - \frac{mbc}{a} \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= m'z - \frac{n'ac}{b} \\ y &= n'z + \frac{m'bc}{a} \end{aligned}\right\},$$

then it will be seen that these equations satisfy the condition of intersection. Hence if we take any three lines in the second system, and suppose them fixed, and make a line intersect them, it must be a line of the first system, and by assuming all possible positions will generate the surface of the hyperboloid; and since consecutive positions of the generating line do not intersect one another, the surface is what is called a *twisted* surface. The surface evidently admits a second mode of generation, in which a

line belonging in every position to the second system, moves along three fixed lines of the first system.

75. The projections of the generating lines of the hyperboloid upon the principal planes, are tangents to the traces of the surface on those planes.

Suppose $x = mz + h$ to be the equation to a line touching the principal section on zx whose equation is

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1;$$

then
$$\frac{(mz+h)^2}{a^2} - \frac{z^2}{c^2} = 1,$$

must give two equal values for z, and

$$\therefore \frac{z^2}{c^2} - \frac{(mz+h)^2}{a^2} + 1 \text{ must be a perfect square;}$$

$$\therefore \left(\frac{1}{c^2} - \frac{m^2}{a^2}\right)\left(1 - \frac{h^2}{a^2}\right) = \frac{m^2h^2}{a^4}, \quad \text{or } h^2 = a^2 - m^2c^2;$$

$$\therefore x = mz \pm \sqrt{a^2 - m^2c^2},$$

which coincides with the equations to the projections of the generating lines on zx. Similarly, the projections on the other principal planes may be shewn to be tangents to the traces on those planes.

Cor. If through the origin we draw a line parallel to the generating lines, its equations will be $x = mz$, $y = nz$; and the equation to the surface generated by it, eliminating m and n by means of the equation which connects those quantities, will be

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$

which represents the conical asymptote; hence any line drawn through the center parallel to a generating line of the hyperboloid lies in the surface of the conical asymptote.

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76. The hyperbolic paraboloid can have an infinite number of straight lines entirely coinciding with its surface.

Let the equations to the surface and to a straight line be respectively

$$x = \frac{y^2}{l} - \frac{z^2}{l'},$$

$$x = mz + h, \quad y = nz + k;$$

then, if the line coincide with the surface in all its points, the equation

$$mz + h = \frac{(nz + k)^2}{l} - \frac{z^2}{l'}$$

is satisfied for all values of z;

$$\therefore \frac{n^2}{l} - \frac{1}{l'} = 0, \quad m = \frac{2nk}{l}, \quad h = \frac{k^2}{l};$$

$$\therefore n = \pm \sqrt{\frac{l}{l'}}, \quad k = \frac{lm}{2n}, \quad h = \frac{l'm^2}{4};$$

therefore the two lines whose equations are respectively

$$\left.\begin{aligned} x &= m\left(z + \frac{ml'}{4}\right) \\ y &= n\left(z + \frac{ml'}{2}\right) \end{aligned}\right\} \ldots(1), \qquad \left.\begin{aligned} x &= m\left(z + \frac{ml'}{4}\right) \\ y &= -n\left(z + \frac{ml'}{2}\right) \end{aligned}\right\} \ldots(2),$$

will coincide with the surface of the paraboloid in all their points, m being any quantity, and $n = \sqrt{\frac{l}{l'}}$; and if m be made to assume all values, we shall determine two infinite systems of lines represented by equations (1) and (2), having the aforesaid property.

Cor. In the same manner as for the hyperboloid, it may be shewn that two lines in the same system never intersect, and that two lines in different systems always intersect. Hence if we suppose three lines in either system to become fixed, the surface may be generated by making a line move

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so as always to intersect them. Or since the generating lines in the two systems are respectively parallel to the fixed planes $y = nz$, $y = -nz$, the paraboloid may be also generated in two ways by a straight line which moves so as always to intersect two fixed lines, and to be parallel to a fixed plane.

In this case also the projections of the generating lines are tangents to the principal sections of the surface.

Thus (changing the sign of x in both equations),

$-x = mz + h$ will be a tangent to $z^2 = l'x$,

if $z^2 + l'mz + l'h = 0$ be a perfect square,

$$\text{or } 4l'h = l'^2m^2, \quad \text{or } h = \frac{m^2l'}{4};$$

$$\therefore -x = m\left(z + \frac{ml'}{4}\right) \text{ is the equation to a tangent;}$$

which, measuring x in the positive direction, coincides with the equation to the projection of the generating lines on zx.

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77. The following Problems furnish examples of finding the equations to surfaces from the given geometrical mode of describing them, and serve to illustrate the preceding propositions. As the simplicity of the result will in every case greatly depend upon properly selecting the positions of the co-ordinate axes, some instances are here introduced with that special object.

PROB. 1. If a straight line have always three given points in three fixed planes at right angles to one another, to find the locus of any other point in it.

Let the planes be taken for the co-ordinate planes, and let the line meet them in the points A, B, C, (fig. 23.); and let P be the describing point the co-ordinates of which are $ON = x$, $NQ = y$, $PQ = z$. Also let $PA = a$, $PB = b$,

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$PC = c$, $\angle ACa = i$, $\angle Cbx = \theta$, Ca being the projection of CA on xy. Then

$$z = c \sin i, \quad x = a \cos i \cos\theta, \quad y = b \cos i \sin\theta;$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\cos i)^2 + (\sin i)^2 = 1;$$

hence the locus of P is an ellipsoid.

PROB. 2. To find the locus of the intersection of two planes drawn through two given lines, so as always to be perpendicular to one another.

Let AA', BB', (fig. 24.) be the two given lines, AB their shortest distance which make the axis of z; also take the plane bisecting AB at right angles for that of xy, and the line bisecting the angle between the projections Oa, Ob of the lines, for the axis of x. Then if $OA = OB = c$, and $\tan aOx = m$, the equations to the lines AA', BB', are $y = mx$, $z = c$; $y = -mx$, $z = -c$.

Let $z = Ax + By + c$ be the equation to the plane passing through AA'; $\therefore A + Bm = 0$;

$$\therefore z - c = A\left(x - \frac{y}{m}\right) \text{ or } A = \frac{m(z-c)}{mx - y}, \quad B = -\frac{z-c}{mx-y}.$$

Similarly, if $z = A'x + B'y - c$ be the equation to the plane passing through BB', $A' - B'm = 0$;

$$\therefore A' = \frac{m(z+c)}{mx+y}, \quad B' = \frac{z+c}{mx+y}.$$

But, because the planes are at right angles to one another,

$$AA' + BB' + 1 = 0;$$

$$\therefore \frac{m^2(z^2-c^2)}{(mx)^2 - y^2} - \frac{z^2-c^2}{(mx)^2 - y^2} + 1 = 0,$$

$$\text{or } (m^2-1)z^2 + m^2x^2 - y^2 = (m^2-1)c^2,$$

which, since only one of the coefficients is negative whether $m >$ or < 1, represents a hyperboloid of one sheet.

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Cor. It may be easily shewn that each of the lines AA', BB', lies in the surface, and that a plane perpendicular to either of them will cut it in a circle. If the lines are parallel, $m = 0$, and $y^2 + z^2 = c^2$, which represents a cylinder as it ought.

Prob. 3. To find the locus of a point which is equidistant from two lines given in space.

Retaining the same axes and notation as in Prob. 2, it will be found that the equation to the surface is

$$mxy + (1 + m^2)cz = 0.$$

Prob. 4. Two points move in straight lines with uniform velocities, to find the equation to the surface generated by the straight line which joins them.

Retaining the same axes and notation as in Prob. 2, and besides denoting by a, a', the initial values of x for the points moving in AA', BB', respectively, and by $n : 1$ the ratio of their velocities, it will be found that the equation to the surface is

$$(n+1)c(cy - mxz) - (n-1)c(mcx - zy) = m(a - na')(c^2 - z^2).$$

Prob. 5. To find the equation to the surface generated by a straight line, which constantly passes through three fixed straight lines.

Let a parallelopiped be constructed, having its three edges (of which no two either intersect or are parallel to one another) in the three given lines; take its center for origin and lines parallel to its edges for the axes; then the equations to the given lines will be

$$\left.\begin{matrix} x = -a \\ z = c \end{matrix}\right\}, \quad \left.\begin{matrix} y = b \\ z = -c \end{matrix}\right\}, \quad \left.\begin{matrix} x = a \\ y = -b \end{matrix}\right\}.$$

Let the equations to the generating line in any position be

$$x = mz + h, \quad y = nz + k, \quad \frac{x-h}{m} = \frac{y-k}{n};$$

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then since it meets each of the given lines, we have three equations of condition, viz.

$$-a = mc + h, \quad b = -nc + k, \quad \frac{a-h}{m} = -\frac{b+k}{n};$$

and it remains to eliminate m, n, h, k, between these and the equations to the generating line. Subtracting we find

$$x + a = m(z - c), \quad y - b = n(z + c), \quad \frac{x-a}{m} = \frac{y+b}{n};$$

therefore, eliminating m and n,

$$(x + a)(y + b)(z + c) = (x - a)(y - b)(z - c),$$

or reducing

$$ayz + bzx + cxy + abc = 0.$$

Cor. If the signs of the quantities a, b, c, be changed this equation is not altered; therefore the same surface would be generated by a straight line constantly passing through three different straight lines whose equations are

$$\left.\begin{matrix} x = a \\ z = -c \end{matrix}\right\}, \quad \left.\begin{matrix} y = -b \\ z = c \end{matrix}\right\}, \quad \left.\begin{matrix} x = -a \\ y = b \end{matrix}\right\}.$$

The surface, as we know, is a hyperboloid of one sheet; the origin is its center, and the axes of the co-ordinates are situated in the surface of the conical asymptote.

Prob. 6. To find the surface generated by a line which always intersects two given lines, and is parallel to a fixed plane.

Let the fixed plane be taken for that of yz, and let AB (fig. 24.), the line joining the traces of the given lines on that plane, be taken for the axis of z, and its middle point for the origin. Also, let the plane of xy be parallel to the two given lines, and the axis of x bisect the angle between their projections made parallel to Oz.

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Then, referred to this system of oblique axes, the equations to the given lines are

$$\left.\begin{aligned} y &= mx \\ z &= c \end{aligned}\right\}, \qquad \left.\begin{aligned} y &= -mx \\ z &= -c \end{aligned}\right\},$$

Let $x = a$, $y = nz + b$, be the equations to the generating line, since it is parallel to yz; then since it intersects each of the given lines,

$$ma = nc + b, \quad -ma = -nc + b;$$

$$\therefore\ b = 0, \text{ and } ma = nc, \text{ or eliminating } a \text{ and } n,\ mx = \frac{y}{z}c;$$

$$\therefore\ mzx = cy \text{ is the required equation,}$$

representing, as we know, a hyperbolic paraboloid.

PROB. 7. If the axes of a hyperboloid become evanescent, always preserving a constant ratio to one another, the surface is changed into its asymptotic cone.

Let the equation to the hyperboloid be

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

a, b, c being certain given values of the semiaxes; and suppose the semiaxes to *diminish* continually, but always to be proportional to a, b, c; then they may be represented by na, nb, nc, where n is a common multiplier continually growing less; therefore the equation to the surface in its successive changes will be

$$\frac{x^2}{n^2a^2} \pm \frac{y^2}{n^2b^2} - \frac{z^2}{n^2c^2} = 1,$$

$$\text{or,}\quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = n^2;$$

therefore, making $n = 0$, or the semiaxes na, nb, nc evanescent, the equation to the surface becomes

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

the same as that to its conical asymptote.

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SECTION III.

ON THE PROJECTIONS OF LINES AND PLANE SURFACES, AND ON THE TRANSFORMATION OF CO-ORDINATES.

78. THE meaning of the projection of a point, and of a line, upon any plane has already been explained, (Arts. 2. and 13.) Moreover if from the extremities of a limited line we drop perpendiculars upon any indefinite line either in the same plane with it or not, the part of the latter line intercepted between the feet of the perpendiculars, is called the projection of the former line upon the latter. Also, if the sides of any plane surface be projected upon a plane, the figure bounded by these projections is called the projection of the given surface upon that plane. Between the length of a finite line, and the length of its projection upon any plane or line, and also between the area of any plane surface, and the area of its projection upon a plane, a remarkable relation exists, which we shall now exhibit.

79. The length of the projection of a limited line upon a plane, is equal to the length of the line multiplied by the cosine of the acute angle which it forms with the plane.

Let CD (fig. 25.) be the line produced to meet GdK, the plane on which it is to be projected, in H. Let DHd be the projecting plane, intersecting GdK in Hd; and in the projecting plane draw Cc, Dd, perpendicular to Hd, and CD' parallel to it; then cd is the projection of CD, and

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$\angle DHd = i$ is the inclination of CD to the plane GdK; also $CD' = CD \cos i$;

$$\therefore cd = CD' = CD \cos i.$$

80. The length of the projection of a limited line upon any other line, is equal to the length of the line multiplied by the cosine of the acute angle which they form with one another.

Let CD (fig. 26.) be the line, AB the line upon which it is to be projected, Cc, Dd perpendiculars upon AB, then cd is the projection of CD upon AB. Let Mc, Nd be planes through C, D, perpendicular to AB, and of course containing the lines Cc, Dd; draw CD' parallel to AB, meeting the plane Nd in D', and join DD'; then the triangle CDD' is right-angled at D', and $\angle DCD' = i$ is equal to the angle formed by the lines, $\therefore CD' = CD \cos i$, but $cd = CD'$, each being the perpendicular distance of parallel planes;

$$\therefore cd = CD \cos i.$$

81. The area of the projection of any plane surface upon a plane, is equal to the area of the surface multiplied by the cosine of the acute angle which the planes form with one another.

Let ACB (fig. 27.) be a triangular area traced in a plane which intersects the plane GdK, upon which the projection is to be made, in GK. Through the angular points draw planes AGa, BKb, CHc perpendicular to GK; then these planes will contain the perpendiculars Aa, Bb, Cc let fall from the angular points upon the plane of projection, and their intersections with each of the planes AHB, aHb, will be perpendicular to GK; also the angle CHc will equal the inclination of the planes $= i$. Join the feet of the perpendiculars; then acb is the projection of ACB; also join Dd, which is parallel to Cc.

Then the triangles acd, ACD, since they have a common altitude GH, are to one another as their bases cd

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CD; similarly the triangles bcd, BCD, having a common altitude HK, are to one another as cd to CD;

$$\therefore \text{ triangular area } abc = \text{triangular area } ABC \cdot \frac{cd}{CD},$$

$$= \text{trianglar area } ABC \cdot \cos i.$$

Next suppose the figure to be projected is a polygon; then it can be divided into triangles, each of which will be to its projection as 1 to $\cos i$; and therefore the sum of the triangles will have to the sum of the projections the same ratio,

$$\text{or area of projection} = \text{area of polygon} \cdot \cos i;$$

and as this is true however the number of sides of the polygon be increased, it is also true when the figure to be projected is bounded by a curve;

$\therefore$ area of projection of any plane surface = area of surface $\cdot \cos i$.

82. The square of the area of any plane surface is equal to the sum of the squares of the areas of its projections on three co-ordinate rectangular planes.

Let the given area be denoted by A, and its projections on the planes of xy, yz, zx by A_z, A_x, A_y, respectively. Also let α, β, γ denote the angles which a perpendicular to the plane of the given area from the origin, makes with the axes of x, y, z. Then γ (Art. 27.) is the inclination of the plane of the given area to the plane of xy, and therefore $A_z = A \cos\gamma$; similarly $A_x = A \cos\alpha$, $A_y = A \cos\beta$;

$$\therefore A_z^2 + A_x^2 + A_y^2 = A^2\{(\cos\gamma)^2 + (\cos\alpha)^2 + (\cos\beta)^2\} = A^2.$$

Cor. Hence in fig. 9, since the triangles AOB, BOC, COA, are the three projections of the triangular area ABC, if OA, OB, OC be denoted by a, b, c,

$$(\Delta ABC)^2 = (\tfrac{1}{2}ab)^2 + (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ca)^2;$$

$$\therefore \Delta ABC = \tfrac{1}{2}\sqrt{(ab)^2 + (ac)^2 + (bc)^2}.$$

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83. To express the distance of a point from the origin in terms of its oblique co-ordinates.

In fig. 2, let the angles yOx, zOx, xOy, be denoted by λ, μ, ν; and let OA, AN parallel to Oy, and NM parallel to Oz, be the co-ordinates x, y, z of the point M. From the points A and N drop perpendiculars Am, Nn upon Ox; then mn is the projection of AN upon Ox, and $= AN\cos\lambda$, since AN is parallel to Oy (Art. 80.); and $Om = OA\cos\mu$. Now from the triangle ONM, since MN is parallel to Oz, we have

$$OM^2 = ON^2 + NM^2 + 2MN \cdot NO\cos zON.$$

But $ON^2 = x^2 + y^2 + 2xy\cos\nu$, from the triangle OAN,

and $ON\cos zON = On = Om + mn = x\cos\mu + y\cos\lambda$;

$$\therefore\ d^2 = x^2 + y^2 + z^2 + 2xy\cos\nu + 2xz\cos\mu + 2yz\cos\lambda.$$

Cor. Since d is the diagonal of the parallelopiped of which x, y, z, are three conterminous edges, the above formula gives the diagonal of any parallelopiped in terms of its edges and the angles which they make with one another. It is also the equation to the surface of a sphere whose center is at the origin and radius $= d$.

84. To find the distance between two points in terms of their oblique co-ordinates.

Let x, y, z; x', y', z', denote the co-ordinates of two points referred to oblique axes; then as in Art. 6, if through these points we draw six planes parallel to the co-ordinate planes, we shall form a parallelopiped with its edges parallel to the axes, their lengths being $x' - x$, $y' - y$, and $z' - z$; and the distance of the points is the diagonal of this parallelopiped;

$$\therefore\ d^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$$
$$+2(x' - x)(y' - y)\cos\nu + 2(x' - x)(z' - z)\cos\mu + 2(y' - y)(z' - z)\cos\lambda.$$

Cor. Hence we have the equation to the surface of a sphere whose radius is d, and co-ordinates of its center x', y', z'.

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85. To find the angle between two lines, whose equations are given referred to oblique axes.

Let $\left.\begin{matrix} x = mz \\ y = nz \end{matrix}\right\}$ $\left.\begin{matrix} x = m'z \\ y = n'z \end{matrix}\right\}$ be the equations to two lines parallel to the proposed ones, through the origin.

Then, as in Art. 30, if we take a point in each of the lines, at a distance $= 1$ from the origin, and call their coordinates x, y, z; x', y', z', respectively; also their distance d, and the angle between the lines θ, we have

$$2 - 2\cos\theta = d^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2 + 2(x'-x)(y'-y)\cos\nu$$
$$+ 2(x'-x)(z'-z)\cos\mu + 2(y'-y)(z'-z)\cos\lambda,$$

or, since

$$1 = x^2 + y^2 + z^2 + 2xy\cos\nu + 2xz\cos\mu + 2yz\cos\lambda,$$

$$1 = x'^2 + y'^2 + z'^2 + 2x'y'\cos\nu + 2x'z'\cos\mu + 2y'z'\cos\lambda,$$

$$\cos\theta = xx' + yy' + zz' + (x'y + xy')\cos\nu + (x'z + xz')\cos\mu$$
$$+ (y'z + z'y)\cos\lambda.$$

But since $x = mz$, $y = nz$, $x' = m'z'$, $y' = n'z'$,

$$1 = z^2(1 + m^2 + n^2 + 2mn\cos\nu + 2m\cos\mu + 2n\cos\lambda),$$

$$1 = z'^2(1 + m'^2 + n'^2 + 2m'n'\cos\nu + 2m'\cos\mu + 2n'\cos\lambda);$$

$$\therefore \cos\theta =$$

$$\frac{1 + mm' + nn' + (m'n + mn')\cos\nu + (m + m')\cos\mu + (n + n')\cos\lambda}{\sqrt{1 + m^2 + n^2 + 2mn\cos\nu + 2m\cos\mu + 2n\cos\lambda}\sqrt{1 + m'^2 + n'^2 + 2m'n'\cos\nu + 2m'\cos\mu + 2n'\cos\lambda}}.$$

Cor. Hence the condition that the lines may be at right angles to one another is $\cos\theta = 0$, or

$$1 + mm' + nn' + (m'n + mn')\cos\nu + (m + m')\cos\mu$$
$$+ (n + n')\cos\lambda = 0.$$

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86. To find the conditions in order that a straight line and a plane, referred to oblique axes, may be perpendicular to one another.

Let a line and plane be drawn through the origin, respectively parallel to the proposed ones, and let their equations be

$$\left.\begin{aligned} x &= mz \\ y &= nz \end{aligned}\right\}, \text{ and } z = Ax + By.$$

Let $\left.\begin{aligned} x &= m'z \\ y &= n'z \end{aligned}\right\}$ be the equations to any line in the plane,

$$\therefore\ 1 = Am' + Bn'.$$

Now the proposed line, since it is perpendicular to the plane, is perpendicular to any line situated in the plane,

$$\therefore\ 1 + mm' + nn' + (mn' + m'n)\cos\nu + (m + m')\cos\mu$$
$$+ (n + n')\cos\lambda = 0.$$

Hence, eliminating m' between these equations,

$$A\{1 + nn' + mn'\cos\nu + m\cos\mu + (n + n')\cos\lambda\}$$
$$+ (m + n\cos\nu + \cos\mu)(1 - Bn') = 0;$$

and as this equation must be true for all values of n',

$$\therefore\ A(1 + m\cos\mu + n\cos\lambda) + (m + n\cos\nu + \cos\mu) = 0.$$

Similarly, by eliminating n', we find the other equation of condition,

$$B(1 + m\cos\mu + n\cos\lambda) + (n + m\cos\nu + \cos\lambda) = 0.$$

Cor. Hence, we can find the angle between two planes referred to oblique axes; for the above conditions enable us to find the equations to two lines through the origin respectively perpendicular to them; and knowing the equations to the lines, we can compute the angle between them, that is, the angle between the planes, by Art. 85.

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87. The discussion of the nature and properties of surfaces is much facilitated, when their equations are reduced to the most simple form they are capable of, without being deprived of any of their generality. This simplification is effected by giving a suitable position to the origin, and suitable directions to the axes of the co-ordinates. We shall therefore next proceed to investigate the chief formulæ for transformation of co-ordinates.

88. To change the origin of the co-ordinates without altering the directions of the axes.

Let x, y, z, be the co-ordinates of the point M (fig. 3) referred to the origin O, and axes Ox, Oy, Oz; also let M' be the new origin, $OA' = h$, $A'N' = k$, $N'M' = l$, its co-ordinates, and x', y', z', the co-ordinates of M referred to M' as origin and axes parallel to the original ones;

$$\text{then } x = OA = M'H + OA' = x' + h,$$

$$\text{similarly } \quad y = y' + k, \quad z = z' + l;$$

and substituting these values of x, y, z in the equation to the surface, we shall obtain the equation referred to the new origin and axes.

89. To pass from one system of co-ordinates to another having the same origin, supposing the first rectangular, and the second oblique.

Let Ox, Oy, Oz (fig. 28.) be the rectangular, and Ox', Oy', Oz' the oblique axes, having a common origin O. From any point P draw PM, PM', respectively parallel to Oz, Oz', meeting the planes of xy, $x'y'$ in M and M', and draw MN, $M'N'$ parallel to Oy, Oy'.

Then the co-ordinates of P in the two systems are

$$ON = x, \quad NM = y, \quad MP = z,$$

$$ON' = x', \quad N'M' = y', \quad M'P = z';$$

and our object is to express each of the first set in terms of the latter. From N', M', drop perpendiculars $N'n$,

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$M'm$ upon Ox, and join PN which is also perpendicular to Ox; then On, nm, mN are the projections of ON', $N'M'$, $M'P$ upon the axis of x; therefore (Art. 80.)

$$On = x' \cos x'x, \quad nm = y' \cos y'x, \quad mN = z' \cos z'x,$$

denoting, by $x'x$, the $\angle x'Ox$ contained by the axes of x' and x produced in the positive directions, and similarly of the others.

$$\therefore x = ON = On + nm + mN$$

$$ON = X = x' \cos x'x + y' \cos y'x + z' \cos z'x; \quad (1.) \quad \text{Rem.}$$

that is, *each primitive co-ordinate is equal to the sum of the projections of the three new co-ordinates upon its axis.*

Hence $y = x' \cos x'y + y' \cos y'y + z' \cos z'y,$

$$z = x' \cos x'z + y' \cos y'z + z' \cos z'z;$$

or, if we denote by (m, n, r) the cosines of the angles which the axes of x', y', z', make with the axis of x; and by (m', n', r'), (m'', n'', r'') similar quantities relative to the axes of y, and z, we have

$$x = mx' + ny' + rz',$$
$$y = m'x' + n'y' + r'z',$$
$$z = m''x' + n''y' + r''z'.$$

Of the nine angles involved in these formulæ, six only are independent, there being three equations of condition; for since $x'x$, $x'y$, $x'z$, are the angles which a straight line, viz. the axis of x', makes with the three rectangular axes of x, y, z,

$$\therefore (\cos x'x)^2 + (\cos x'y)^2 + (\cos x'z)^2 = 1;$$

and similarly with respect to the angles which the axes of y' and z' make with the primitive axes; therefore

$$\left.\begin{aligned} m^2 + m'^2 + m''^2 &= 1, \\ n^2 + n'^2 + n''^2 &= 1, \\ r^2 + r'^2 + r''^2 &= 1. \end{aligned}\right\} \ldots (2).$$

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Cor. In the figure, the axes of x', y', z' are supposed to make acute angles with the axis of x; if this were not the case, and if for instance one of them Oy', made an obtuse angle with Ox, m would fall to the left of n, and we should have

$$x = On - nm + mN,$$

with which, the general expression (1) still agrees, because the term $y' \cos y'x$ is negative in the case supposed, but numerically equal to the projection of y'. If again, the axes being as in the figure, y' was negative, the point m would fall to the left of n, and we should have

$$x = On - nm + mN.$$

Hence, we collect that the formula (1) is applicable to all cases, provided we pay attention to the signs of the co-ordinates, and of the cosines; the angles, as was before observed, being those formed by the axes produced in the positive directions.

90. To pass from one system of rectangular co-ordinates to another also rectangular.

Here it would be sufficient to join to the expressions and equations of condition of the preceding Art., three new equations of condition expressing that the axes Ox', Oy', Oz' are at right angles to one another.

We may, however, arrive at the expressions for x, y, z, briefly as follows. Let d be the distance of P from the origin; then referring the lines OP, Ox to the rectangular axes Ox', Oy', Oz', we have for the angle between them

$$\cos Px = \cos Px' \cos xx' + \cos Py' \cos xy' + \cos Pz' \cos xz';$$

hence, multiplying by d, and observing that $d \cos Px = x$, $d \cos Px' = x'$, &c., we have

$$x = x' \cos x'x + y' \cos y'x + z' \cos z'x,$$

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and similarly for y and z; hence denoting the cosines as before,

$$\left.\begin{aligned} x &= mx' + ny' + rz', \\ y &= m'x' + n'y' + r'z', \\ z &= m''x' + n''y' + r''z'. \end{aligned}\right\} \ldots (3).$$

In this case there are only three independent angles. For since the axes of x' and y' are at right angles,

$$\cos x'x \cos y'x + \cos x'y \cos y'y + \cos x'z \cos y'z = 0;$$

similarly, expressing that the axes of x', z', and the axes of y', z', are at right angles, and substituting for the cosines their values, we have three new equations of condition to be joined to those of the preceding Article, viz.

$$\left.\begin{aligned} mn + m'n' + m''n'' &= 0, \\ mr + m'r' + m''r'' &= 0, \\ nr + n'r' + n''r'' &= 0. \end{aligned}\right\} \ldots (4).$$

91. Sometimes, in the case of two rectangular systems, it is required to find x', y', z' in terms of x, y, z. This may be effected by regarding x', y', z' as the primitive co-ordinates, and recollecting that each is equal to the sum of the projections of the co-ordinates x, y, z upon its axis;

$$\therefore\ x' = x \cos xx' + y \cos yx' + z \cos zx';$$

and similarly for y' and z'; hence

$$x' = mx + m'y + m''z,$$

$$y' = nx + n'y + n''z,$$

$$z' = rx + r'y + r''z.$$

These expressions might also have been obtained from (3) by adding them together after having multiplied them respectively (1) by m, m', m''; (2) by n, n', n''; (3) by r, r', r''; reducing in each case by means of the equations of condition (2) and (4).

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Also the equations of condition, when x, y, z are regarded as the new co-ordinates, and consequently their axes referred to the axes of x', y', z', will be

$$\left.\begin{aligned} m^2 + n^2 + r^2 &= 1, \\ m'^2 + n'^2 + r'^2 &= 1, \\ m''^2 + n''^2 + r''^2 &= 1. \end{aligned}\right\} \ldots(5),$$

$$\left.\begin{aligned} mm' + nn' + rr' &= 0, \\ mm'' + nn'' + rr'' &= 0, \\ m'm'' + n'n'' + r'r'' &= 0. \end{aligned}\right\} \ldots(6),$$

which are entirely equivalent to the relations between the same constants obtained before, and may in every case replace them. Indeed there is no difficulty in shewing that they are a necessary consequence of (2) and (4). For since P is at the same distance from the origin in either system,

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2;$$

and putting for x, y, z their values given in (3), and equating coefficients on both sides, we obtain the equations of condition (5) and (6).

92. In the preceding Articles, the means of passing from one rectangular system of co-ordinates to another, have been given in simple and symmetrical formulæ; they have however the inconvenience of involving nine constants, six of which must be eliminated by means of six equations of condition in order to make the determination of the new axes relative to the primitive ones depend upon three quantities. This led Euler to invent a mode of expressing each of the nine constants in terms of three others, which are *the inclination of the planes of* xy *and* x'y', *and the angles which their line of intersection forms with the axes of* x *and* x'; as described in the following proposition.

93. To pass from the rectangular system of co-ordinates x, y, z, to another rectangular system x', y', z'; having given the $\angle\theta$ at which the planes of xy, $x'y'$, are inclined

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to one another, and the angles ϕ, ψ, which their line of intersection makes with the axes of x and x' respectively.

Let the new plane of $x'y'$ intersect the plane of xy in the line Ox_1 (fig. 29.), and be situated above it; and let Oz' perpendicular to the plane $x'y'$ be the axis of z', and Ox', Oy', the axes of x' and y'; and let a sphere be described about O as a centre with radius 1, cutting the axes in the points x, y, z, &c.; and suppose these points to be joined by arcs of great circles. Then if $xx_1 = \phi$, and $\angle x'x_1y = \theta$, or $zz' = \theta$, the plane of $x'y'$ and the axis Oz' will be completely determined; also if $x'x_1 = \psi$, then the positions of Ox', Oy', in the plane x_1Oy' will be fixed.

Also we have

$$x = x' \cos x'x + y' \cos y'x + z' \cos z'x, \text{ &c.};$$

and our object is to express the nine cosines in terms of θ, ϕ, ψ, which may be done by means of the fundamental theorem in Spherical Trigonometry for finding one side of a triangle in terms of the other two and the included angle.

First, from the triangle $x'x_1x$, in which $\angle x'x_1x = 180^0 - \theta$,

$$\cos x'x = \cos\phi \cos\psi - \sin\phi \sin\psi \cos\theta;$$

and changing ψ into $90^0 + \psi$,

$$\cos y'x = -\cos\phi \sin\psi - \sin\phi \cos\psi \cos\theta;$$

also from the triangle $z'x_1x$, in which

$$z'x_1 = 90^0, \quad \text{and} \quad z'x_1x = 90^0 - \theta,$$

$$\cos z'x = \sin\phi \sin\theta.$$

Secondly, from the triangle $x'x_1y$, in which $x_1y = 90^0 - \phi$,

$$\cos x'y = \cos\psi \sin\phi + \sin\psi \cos\phi \cos\theta;$$

and changing ψ into $90^0 + \psi$,

$$\cos y'y = -\sin\psi \sin\phi + \cos\psi \cos\phi \cos\theta;$$

also from the triangle $z'x_1y$, in which

$$z'x_1 = 90^0, \quad \text{and} \quad z'x_1y = 90^0 + \theta,$$

$$\cos z'y = -\cos\phi \sin\theta.$$

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Lastly, from the triangle $x'x_1z$, in which $zx_1 = 90^0$, and $x'x_1z = 90^0 - \theta$,

$$\left.\begin{array}{l} \\ \cos x'z = \sin\psi \sin\theta; \end{array}\right.$$

and changing ψ into $90^0 + \psi$,

$$\cos y'z = \cos\psi \sin\theta;$$

$$\text{also} \quad \cos z'z = \cos\theta;$$

Euler's equations for Linkage: Ref. p. 8.

$$\left.\begin{array}{l} \therefore\ x = x'(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) \\ \quad - y'(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) \\ \quad + z'\sin\phi\sin\theta. \\ y = x'(\cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta) \\ \quad - y'(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta) \\ \quad - z'\cos\phi\sin\theta. \\ z = x'\sin\psi\sin\theta + y'\cos\psi\sin\theta + z'\cos\theta. \end{array}\right.$$

94. We may arrive at these results without the aid of Spherical Trigonometry, by passing successively through three rectangular systems, each having one axis in common with the preceding, and employing the formulæ realtive to the transformation of co-ordinates in one plane. Thus to pass from the system of axes Ox, Oy, Oz, to the system Ox_1, Oy_1, Oz, of which Ox_1 is the trace of the plane $x'Oy'$ upon xy, and Oy_1 is perpendicular to Ox_1 in that plane, we must put

See Linkage's work p. 8

$$\left.\begin{array}{l} x = x_1\cos\phi - y_1\sin\phi \\ y = x_1\sin\phi + y_1\cos\phi \end{array}\right\} \ldots (1.)$$

without altering z. Next to pass from the system Ox_1, Oy_1, Oz, to the system Ox_1, Oy_2, Oz', of which Oy_2 lies in the plane $x'y'$, and Oz' is perpendicular to Oy_2 in the plane y_1Oz, we must put

$$\left.\begin{array}{l} y_1 = y_2\cos\theta - z'\sin\theta \\ z = y_2\sin\theta + z'\cos\theta \end{array}\right\} \ldots (2.)$$

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without altering x_1. Lastly, to pass from the system Ox_1, Oy_2, Oz' to the system Ox', Oy', Oz', of which Ox', Oy', are at right angles to one another in the plane $x_1 Oy_2$, we must put

$$\left.\begin{aligned} x_1 &= x' \cos\psi - y' \sin\psi \\ y_2 &= x' \sin\psi + y' \cos\psi \end{aligned}\right\} \ldots (3.)$$

without altering z'. But we may arrive at the result of these three successive substitutions by a single substitution, the formulæ for which will be formed by eliminating x_1, y_1, and y_2 between the systems of equations (1), (2) and (3); we shall then obtain the same expressions for x, y, z in terms of x', y', z' and the three angles, as before.

Cor. If we dispense with the third transformation, we shall have Ox_1, one of the new axes, in the plane of xy. The system Ox_1, Oy_2, Oz', in which the new plane of xy is inclined at an $\angle\theta$ to the primitive one, and their intersection is the new axis of x inclined at an $\angle\phi$ to the primitive one, is for most purposes sufficiently general. The formulæ for it, combining the equations (1) and (2), and calling the new co-ordinates x', y', z', are

$$x = x' \cos\phi - (y' \cos\theta - z' \sin\theta) \sin\phi,$$

$$y = x' \sin\phi + (y' \cos\theta - z' \sin\theta) \cos\phi,$$

$$z = y' \sin\theta + z' \cos\theta;$$

which evidently agree with the formulæ of Art. 93, when we put $\psi = 0$.

95. In the preceding transformations we have supposed the origin to remain unaltered; if, however, the origin is to be changed, as well as the directions of the axes, we must employ the formulæ

$$x = x'' + h, \quad y = y'' + k, \quad z = z'' + l,$$

where h, k, l are the co-ordinates of the new origin parallel to the primitive axes, and x'', y'', z'' denote the values of x, y, z found in each of the preceding cases.

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96. It is important to observe, that in employing any of the preceding formulæ to refer an equation $F(x, y, z) = 0$ to new axes, the transformed equation $F(x', y', z') = 0$ will always be of the same degree as the primitive equation; understanding by the degree of an equation the sum of the indices of the three variables in that term where it is greatest. For let this term be $Ax^p y^q z^t$; then it becomes by substitution,

$$A(mx' + ny' + rz')^p (m'x' + n'y' + r'z')^q (m''x' + n''y' + r''z')^t;$$

now these factors cannot furnish terms whose dimensions exceed p, q, t, respectively; therefore if $A' x'^{p'} y'^{q'} z'^{t'}$ be the term of highest dimension, $p' + z' + t'$ cannot exceed $p + q + t$; *i. e.* the dimension of the transformed equation cannot be greater than that of the primitive. Neither can it be less; for if it could, then, as transformation of co-ordinates can never raise the degree of an equation, we could not return from $F(x', y', z') = 0$ to $F(x, y, z) = 0$, which is absurd.

97. To pass from one system of oblique co-ordinates to another also oblique.

Let $OM = x$, $MQ = y$, $QP = z$, (fig. 30.) be the co-ordinates of a point P parallel to the axes Ox, Oy, Oz; and $OM' = x'$, $M'Q' = y'$, $Q'P = z'$ the co-ordinates of the same point parallel to the axes Ox', Oy', Oz'. Through the origin draw a normal ON'' to the plane of xy, and on the same side of it as the positive direction of the axis of z; and from P, Q', M' drop perpendiculars Pp, $Q'n$, $M'm$, upon ON'';

$$\text{then } PQ \cos N''Oz = Op = Om + mn + np$$
$$= OM' \cos N''x' + M'Q' \cos N''y' + Q'P \cos N''z',$$
$$\text{or } z \cos N''z = x' \cos N''x' + y' \cos N''y' + z' \cos N''z'.$$

Similarly, if we draw normals ON, ON' to the planes of yz, zx, and on the same side of them as the positive directions of the axes of x and y, we shall have

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$$x \cos Nx = x' \cos Nx' + y' \cos Ny' + z' \cos Nz',$$

$$y \cos N'y = x' \cos N'x' + y' \cos N'y' + z' \cos N'z'.$$

Since $\cos N''z = \sin(z, xy)$, &c. these formulæ may be expressed without the aid of the auxiliary normals; but they are more convenient in their present shape, as the employment of negative angles is thereby avoided.

98. The position of a point P' in space may be fixed by the following three variables; (1) the radius vector $OP' = r$ (fig. 16.); (2) the $\angle COP' = \theta$ which the radius vector makes with the axis of z; (3) the $\angle AOA' = \phi$ which the meridian plane COP' makes with the fixed plane COA. Of these angles the former varies from 0 to 180°, and the latter from 0 to 360°, in order that the radius vector may pass through all points of space. If x, y, z be the co-ordinates of P', they can be expressed by means of r, θ, ϕ; for $P'M = r \cos\theta$, $OM = r \sin\theta$;

$\therefore\ x = OM \cos\phi = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r \cos\theta$.

99. In the discussion of surfaces, it is useful to know the nature and magnitude of the curves in which they are intersected by any planes. To do this, it is not sufficient to combine the equation to the surface $F(x, y, z) = 0$, with the equation to the cutting plane $z = Ax + By + c$, so as to eliminate one of the variables, z for instance; for the result $f(x, y) = 0$ would represent only the *projection* of the curve required, which is not generally of the same nature with the section in space, nor sufficient to determine it. But if we transform the co-ordinates so that the cutting plane may be that of $x'y'$, and then put $z' = 0$ in the resulting equation, we shall determine the trace of the surface on $x'y'$, *i. e.* the curve in which it is intersected by the proposed plane. And it may be here observed that, since the degree of an equation is never altered by the transformation of co-ordinates, if the equation to a surface be of the n^{th} degree, the curve in which it is intersected by a plane cannot be of a higher order than the n^{th}; but it may be of a lower, if by putting $z' = 0$, we cause

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all the terms of the highest dimension to disappear from its equation. If we employ Euler's formulæ for tranformation of co-ordinates (Art. 93.), since z' is to vanish in the final result, we are at liberty to make $z' = 0$ in the values of x, y, z before we substitute them, and so we may obtain the proper substitutions; these however, without being deduced from the general case, may be readily obtained as follows.

100. To determine the nature of the section of a surface made by any plane passing through the origin.

The most convenient data for fixing the position of the cutting plane are (1) the $\angle\theta$ at which it is inclined to the plane of xy, and (2) the $\angle\phi$ which its trace on that plane makes with the axis of x; these are readily obtained if we suppose the equation to the plane given; for if the equation be $z = Ax + By$, then $\cos\theta = \frac{1}{\sqrt{1 + A^2 + B^2}}$ (Art. 29.), and $\tan\phi = -\frac{A}{B}$, since $Ax + By = 0$ is the equation to the trace.

Let $x'Oy'$ (fig. 31.) be the given plane, cutting the surface in the curve AM, and the plane of xy in the line Ox', which take for the axis of x', and let Oy', a line perpendicular to it in the given plane, be the axis of y', and $OR = x'$, $RM = y'$ the co-ordinates of any point M in the section referred to the axes Ox', Oy'; also let $OQ = x$, $QP = y$, $PM = z$ be the co-ordinates of the same point referred to the axes Ox, Oy, Oz. Join RP, then $\angle MRP = \theta$ the inclination of the cutting plane to the plane of xy, and $\angle xOx' = \phi$ the angle formed by its trace on xy with the axis of x.

Then $PR = y'\cos\theta$, $PM = y'\sin\theta$;

$$OQ = OR\cos\phi + RP\sin\phi, \quad QP = OR\sin\phi - RP\cos\phi;$$

$$\therefore z = y'\sin\theta, \quad x = x'\cos\phi + y'\cos\theta\sin\phi,$$

$$y = x'\sin\phi - y'\cos\theta\cos\phi;$$

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and if these values be substituted in the equation $F(x, y, z)=0$ to the surface, we shall have a relation between x' and y' which is the equation to the curve AM.

101. If the cutting plane pass through one of the co-ordinate axes, the formulæ are simplified, and in many cases are sufficiently general.

Let $x'Oy$ (fig. 32.) be the cutting plane, passing through the axis of y; Ox' in the plane of zx the axis of x', $PM = x'$, $OM = y'$, the co-ordinates of any point P in the section, $ON = x$, $NQ = y$, $QP = z$ the co-ordinates of the same point. Join MQ, then $\angle PMQ = \theta$, and $MQ = x' \cos\theta$, $PQ = x' \sin\theta$;

$$\therefore \quad x = x' \cos\theta, \quad y = y', \quad z = x' \sin\theta;$$

that is, to determine the section made by a plane through the axis of y inclined at an $\angle\theta$ to xy, we have only to write $x' \cos\theta$ and $x' \sin\theta$ for x and z, in the equation to the surface, without altering y; and the resulting relation between x' and y is the equation to the curve. Similarly, if the cutting plane pass through the axes of z or x.

102. If the cutting plane does not pass through the origin of the primitive co-ordinates, or if we wish to take a point in the cutting plane different from O for the origin of the new co-ordinates, we shall only have to add to the second members of the preceding formulæ the co-ordinates h, k, l of the new origin reckoned parallel to the primitive axes.

103. To determine the nature of the curve formed by the intersection of any plane with a surface of the second order.

Let $Ax^2 + By^2 + Cz^2 = 2A'x$ be the equation, which may represent all surfaces of the second order; therefore, substituting for x, y, z the values found in Art. 100.

$$x = x' \cos\phi + y' \cos\theta \sin\phi,$$

$$y = x' \sin\phi - y' \cos\theta \cos\phi,$$

$$z = y' \sin\theta,$$

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the result developed and arranged will be

$$x'^2\{A(\cos\phi)^2+B(\sin\phi)^2\}+2x'y'(A-B)\cos\theta\sin\phi\cos\phi$$
$$+y'^2[\{A(\sin\phi)^2+B(\cos\phi)^2\}(\cos\theta)^2+C(\sin\theta)^2]$$
$$=2A'x'\cos\phi+2A'y'\cos\theta\sin\phi,$$

the equation to a curve of the second order, which will be an ellipse, parabola, or hyperbola, according as the quantity

$$(A-B)^2(\cos\theta\sin\phi\cos\phi)^2-\{A(\cos\phi)^2+B(\sin\phi)^2\}$$
$$\times[\{A(\sin\phi)^2+B(\cos\phi)^2\}(\cos\theta)^2+C(\sin\theta)^2],$$
$$\text{or } -AB(\cos\theta)^2-AC(\cos\phi\sin\theta)^2-BC(\sin\phi\sin\theta)^2$$

is negative, nothing, or positive.

Hence every section of an ellipsoid is an ellipse, because all the quantities A, B, C are positive. For a hyperboloid of one or two sheets, in which cases one or two of the quantities A, B, C are negative, the section may be an ellipse, parabola, or hyperbola. For paraboloids $A=0$; therefore for the elliptic, in which case B and C have the same sign, the section is an ellipse; except when $\theta=0$, or $\phi=0$, in which cases it is a parabola. For the hyperbolic paraboloid, since B and C are of contrary signs, the section is a hyperbola; except as before when $\theta=0$, or $\phi=0$, when it is a parabola. For a cone of the second degree every section is a conic. This follows as a particular case.

Cor. Since the section is referred to rectangular axes, it can never be a circle unless the coefficient of $x'y'$ vanishes, or

$$(A-B)\cos\theta\sin\phi\cos\phi=0;$$

hence we must have $\theta=\frac{\pi}{2}$, or $\phi=\frac{\pi}{2}$, or $\phi=0$; which shew that for a circular section, the cutting plane must be perpendicular to one of the principal planes of the surface. In the next article we shall see that this property is confined to one only of the principal planes of each surface.

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104. Every surface of the second order, except the hyperbolic paraboloid, may be generated in two ways by the motion of a variable circle parallel to itself, the center of the circle moving along a diameter of the surface.

Let the surface have a center, and let its equation be

$$Ax^2 + By^2 + Cz^2 = D.$$

Since every circular section must be perpendicular to a principal plane, let the cutting plane be perpendicular to zx, and inclined to xy at an $\angle\theta$; and as the center of the circle must be in the plane of zx, let $x = h$, $z = l$ be its co-ordinates; then the equation to the section reckoned from that point as origin will be

$$A(x' \cos\theta + h)^2 + By'^2 + C(x' \sin\theta + l)^2 = D,$$

(Art. 101.) which represents a circle if

$$A(\cos\theta)^2 + C(\sin\theta)^2 = B, \text{ or } A + C(\tan\theta)^2 = B\{1 + (\tan\theta)^2\},$$

$$\text{or } (\tan\theta)^2 = \frac{B - A}{C - B};$$

and the relation between h and l, since the origin is the center of the circle and therefore the coefficient of $x' = 0$, is

$$A \cos\theta h + C \sin\theta l = 0;$$

therefore the locus of the centers of the circular sections is a straight line and a diameter of the surface.

Also the radius of the section $= \sqrt{\frac{D}{B} - \frac{h^2 + l^2}{B}}$.

We must now examine, for each of the surfaces, which axis it is that coincides with the axis of y to which the cutting plane is parallel.

1. For the ellipsoid, $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$,

$$\therefore \tan\theta = \pm \frac{c}{a}\sqrt{\frac{a^2 - b^2}{b^2 - c^2}};$$

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1. $\therefore$ b lies between a and c, or the axis of the surface to which the cutting plane is parallel is its mean axis.

2. For the hyperboloid of one sheet, since we cannot have B negative, we must put $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = -\frac{1}{c^2}$,

$$\therefore \tan\theta = \pm\frac{c}{a}\sqrt{\frac{b^2 - a^2}{c^2 + b^2}};$$

2. $\therefore$ $b > a$, or the cutting plane is parallel to the greater of the real axes.

3. For the hyperboloid of two sheets, since we cannot have A and C negative, we must put

$$A = \frac{1}{a^2}, \quad B = -\frac{1}{b^2}, \quad C = -\frac{1}{c^2};$$

$$\therefore \tan\theta = \pm\frac{c}{a}\sqrt{\frac{a^2 + b^2}{b^2 - c^2}};$$

3. $\therefore$ $b > c$, or the cutting plane is parallel to the greater of the imaginary axes.

Since $\tan\theta$ has two values, the cutting plane may be inclined at an $\angle\theta$ or $180 - \theta$ to the plane of xy, and hence the surface may be generated by a variable circle in two different ways; but it will be observed that in every case, if the surface become one of revolution, the two positions coincide in one which is parallel to the two equal axes.

105. Next, let the surface not have a center, and let its equation be

$$By^2 + Cz^2 = 2A'x.$$

Then as before, if h, l, denote the co-ordinates of the center of the circular section, its equation reckoned from that point as origin will be

$$By'^2 + C(x'\sin\theta + l)^2 = 2A'(x'\cos\theta + h),$$

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with the conditions

$$B = C(\sin\theta)^2, \quad C\sin\theta l = A'\cos\theta;$$

$$\therefore \sin\theta = \pm\sqrt{\frac{B}{C}}, \quad \text{and } l = \frac{A'}{C}\cot\theta.$$

Hence the locus of the centers of the circular sections is a straight line parallel to the axis of the surface, or a diameter; also B and C have the same sign and $B < C$; hence the surface is the elliptic paraboloid, and the cutting plane is perpendicular to the principal section whose latus rectum is least.

In the case of the hyperbolic paraboloid, since B and C have different signs, no plane can be drawn which shall intersect it in a circle.

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106. We shall now give some applications of the principal results obtained in this section, relative to the projections of plane surfaces, the transformation of co-ordinates, and the intersections of surfaces by planes.

PROB. 1. The pyramid whose vertex is the origin, and base any plane surface, is equal to the three pyramids whose common vertex is any point in the plane of the surface, and bases its three projections on the co-ordinate planes.

Let d, a perpendicular from the origin upon the plane in which the surface lies, make angles α, β, γ with the axes of x, y, z; also let A denote the area of the surface, and x, y, z the co-ordinates of any point in its plane,

$$\therefore x\cos\alpha + y\cos\beta + z\cos\gamma = d.$$

Hence, multiplying by $\frac{1}{3}A$,

$$A\cos\alpha \cdot \tfrac{1}{3}x + A\cos\beta \cdot \tfrac{1}{3}y + A\cos\gamma \cdot \tfrac{1}{3}z = A\frac{d}{3},$$

$$\text{or, } A_x \cdot \frac{x}{3} + A_y \cdot \frac{y}{3} + A_z \cdot \frac{z}{3} = A \cdot \frac{d}{3},$$

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which (since the volume of any pyramid is equal to the area of its base multiplied by one third of its height) expresses the property announced. The quantities A_x, A_y, A_z will be positive or negative according as α, β, γ are acute or obtuse.

PROB. 2. To find the sum of the projections of any number of areas upon a given plane.

Let D', D'', &c. denote the areas; α', β', γ'; α'', β'', γ''; &c. the angles which the perpendiculars upon their planes from the origin make with the rectangular axes of x, y, z produced in the positive directions. Also, let α, β, γ be the angles which a perpendicular to the given plane makes with the same axes. Then the areas must be multiplied respectively by the expressions

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

$$\cos\alpha\cos\alpha'' + \cos\beta\cos\beta'' + \cos\gamma\cos\gamma'', \quad \&c.,$$

which are the values of the cosines of the angles between the plane of projection, and the planes in which the areas lie. The sum of these products will be

$$\begin{aligned} S = {} & \cos\alpha\,(D'\cos\alpha' + D''\cos\alpha'' + \&c.) \\ & + \cos\beta\,(D'\cos\beta' + D''\cos\beta'' + \&c.) \\ & + \cos\gamma\,(D'\cos\gamma' + D''\cos\gamma'' + \&c.); \end{aligned}$$

or, if we denote the sums of the projections on the co-ordinate planes of yz, zx, xy by S_x, S_y, S_z respectively,

$$S = S_x\cos\alpha + S_y\cos\beta + S_z\cos\gamma;$$

which enables us, from knowing the projections of any areas upon three planes at right angles to one another, to find the sum of their projections upon any plane whose position is given relative to the same planes. In forming the values of S_x, S_y, S_z regard must of course be had to the signs of $\cos\alpha'$, $\cos\alpha''$, &c. Thus for a single area D',

$$S = D'_x\cos\alpha + D'_y\cos\beta + D'_z\cos\gamma,$$

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where the quantities D'_x, D'_y, D'_z are positive or negative, according as the angles α', β', γ' are acute or obtuse.

PROB. 3. To find the position of the plane on which the sum of the projections of any number of areas is a maximum.

Let S_x, S_y, S_z denote the sums of the projections of the areas on the co-ordinate planes of yz, zx, xy respectively; $S_{x'}$, $S_{y'}$, $S_{z'}$ the sums of the projections on three other planes $y'z'$, $z'x'$, $x'y'$ at right angles to one another having the same origin,

$$\therefore\ S_{x'} = S_x \cos x'x + S_y \cos x'y + S_z \cos x'z,$$

$$S_{y'} = S_x \cos y'x + S_y \cos y'y + S_z \cos y'z,$$

$$S_{z'} = S_x \cos z'x + S_y \cos z'y + S_z \cos z'z.$$

Hence, squaring and adding, and recollecting the equations of condition to which the cosines are subject,

$$S^2_{x'} + S^2_{y'} + S^2_{z'} = S^2_x + S^2_y + S^2_z.$$

Hence, if the planes of $y'z'$, $z'x'$, $x'y'$, be so situated that the sums of the projections on two of them vanish, the sum of the projections on the third will be the greatest possible; let this be the plane of $y'z'$, then

$$S_{y'} = 0,\quad S_{z'} = 0,\quad \text{and}\quad S^2_{x'} = S^2_x + S^2_y + S^2_z.$$

But if we had begun by supposing the projections on the planes of $y'z'$, $z'x'$, $x'y'$ given, we should have had the equation

$$S_x = S_{x'} \cos xx' + S_{y'} \cos xy' + S_{z'} \cos xz'.$$

In this case therefore, $S_x = S_{x'} \cos xx'$; similarly,

$$S_y = S_{x'} \cos yx',\quad S_z = S_{x'} \cos zx'.$$

Hence the position of the plane (or rather of its normal, viz. the axis of x') on which the sum of the projections is a maximum is determined by the equations

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$$\cos\alpha = \cos x'x = \frac{S_x}{\sqrt{S_x^2 + S_y^2 + S_z^2}},$$

$$\cos\beta = \cos x'y = \frac{S_y}{\sqrt{S_x^2 + S_y^2 + S_z^2}},$$

$$\cos\gamma = \cos x'z = \frac{S_z}{\sqrt{S_x^2 + S_y^2 + S_z^2}};$$

and the value of the greatest sum $S = \sqrt{S_x^2 + S_y^2 + S_z^2}$.

Cor. Let S' denote the sum of the projections on a plane whose normal makes angles α', β', γ' with the axes of x, y, z, and an $\angle = \theta$ with the normal to the plane of greatest projection,

$$\therefore \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

$$= \frac{S_x\cos\alpha' + S_y\cos\beta' + S_z\cos\gamma'}{S} = \frac{S'}{S},$$

$$\therefore S' = S\cos\theta.$$

Hence, the sum of the projections vanishes on all planes perpendicular to that on which the sum is a maximum.

Prob. 4. To find the area of the surface of any portion of a right cone on a circular base.

Let ACM (fig. 33.) be a section through the axis of the cone perpendicular to the cutting plane CPB; $AC = c$, $AB = c'$, $\angle CAB = \beta$; also let Cpb be the projection of CPB on a plane perpendicular to the axis, then $Cb = (c + c')\sin\frac{\beta}{2}$ = major axis of projection; and $\sqrt{cc'}\sin\frac{\beta}{2} = \frac{1}{2}$ minor axis both of CPB and Cpb, since it is a mean proportional between the perpendiculars dropped from B and C upon the axis of the cone;

$$\therefore \text{the area of projection } Cpb = \pi\left(\sin\frac{\beta}{2}\right)^2 \cdot \frac{(c + c')\sqrt{cc'}}{2}.$$

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But Cpb is the projection of the conical surface $CABP$, every portion of which is inclined at an $\angle = \frac{\pi}{2} - \frac{\beta}{2}$ to the plane CM,

$$\therefore \text{ area of surface } CABP = \pi \sin \frac{\beta}{2} \frac{(c + c')\sqrt{cc'}}{2}.$$

PROB. 5. Three straight lines mutually at right angles and meeting in a point, constantly pass through a plane curve of the second order; to find the locus of their point of intersection.

Let $\left.\begin{matrix} Ax^2 + By^2 = C \\ z = 0 \end{matrix}\right\}$ be equations to the plane curve, h, k, l the co-ordinates of the point of intersection of the three lines; and let the curve be referred to that point as origin, and to the three straight lines as axes; let the cosines of the angles formed by the new axes of x', y', z', with the axis of x, be denoted by m, n, r; and similarly for the axes of y and z; therefore by the formula of Art. 90, the transformed equations to the curve are

$$A(mx' + ny' + rz' + h)^2 + B(m'x' + n'y' + r'z' + k)^2 = C,$$

$$m''x' + n''y' + r''z' + l = 0.$$

In order that the axis of x' may meet the curve, these two equations must agree in giving the same value for x', when y' and $z' = 0$;

$$\therefore A(mx' + h)^2 + B(m'x' + k)^2 = C,$$

$$m''x' + l = 0, \quad \text{or } x' = -\frac{l}{m''};$$

hence, substituting this value of x' in the former equation, and proceeding in a similar manner with respect to the axes of y' and z', we have

$$A(m''h - ml)^2 + B(m''k - m'l)^2 = Cm''^2,$$

$$A(n''h - nl)^2 + B(n''k - n'l)^2 = Cn''^2,$$

$$A(r''h - rl)^2 + B(r''k - r'l)^2 = Cr''^2.$$

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Therefore, adding these equations together, and taking account of the equations of condition to which the quantities m, n, &c. are subject,

$$A(h^2 + l^2) + B(k^2 + l^2) = C,$$

the equation to the locus; which is therefore a surface of the second order concentric with the curve.

Cor. If the given curve be an ellipse, this equation becomes, replacing h, k, l by x, y, z,

$$b^2x^2 + a^2y^2 + (a^2 + b^2)z^2 = a^2b^2,$$

that to an ellipsoid; if a hyperbola, the equation to the surface is

$$b^2x^2 - a^2y^2 - (a^2 - b^2)z^2 = a^2b^2,$$

that to a hyperboloid of one or two sheets, according as the transverse axis of the hyperbola $<$ or $>$ the conjugate axis.

In the case of the ellipse, if we remove the origin to the extremity of the major axis, by writing $x - a$ for x, the equations become

$$\frac{x^2}{a^2} - \frac{2x}{a} + \frac{y^2}{b^2} = 0,$$

$$b^2x^2 + a^2y^2 + (a^2 + b^2)z^2 = 2ab^2x;$$

now let $b^2 = 2ap - p^2$, p being the distance of the focus of the ellipse from its vertex, and make a infinite;

$$\therefore y^2 = 4px, \quad y^2 + z^2 = 4px;$$

which shew that when the curve is a parabola, the locus required is the paraboloid which it would generate by revolving about its axis.

Prob. 6. Three straight lines mutually at right angles and meeting in a point are applied to a surface of the second order; to find the locus of their point of intersection.

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Let the equation to the surface be

$$Ax^2 + By^2 + Cz^2 = D;$$

then referring it to the three straight lines as axes and to their point of intersection as origin, the transformed equation is

$$A(mx' + ny' + rz' + h)^2 + B(m'x' + n'y' + r'z' + k)^2$$
$$+ C(m''x' + n''y' + r'z'' + l)^2 = D.$$

Now making successively y', z', $= 0$, x', z', $= 0$, x', y', $= 0$, in order to determine the points where the axes of x', y', z' meet the surface, and calling

$$Ah^2 + Bk^2 + Cl^2 - D = G, \text{ we have}$$

$$x'^2(m^2A + m'^2B + m''^2C) + 2x'(mAh + m'Bk + m''Cl) + G = 0,$$

$$y'^2(n^2A + n'^2B + n''^2C) + 2y'(nAh + n'Bk + n''Cl) + G = 0,$$

$$z'^2(r^2A + r'^2B + r''^2C) + 2z'(rAh + r'Bk + r''Cl) + G = 0.$$

But since the axes of x', y', z' touch the surface, the two points in which any axis meets it coincide in one; therefore the roots of each of the above equations are equal, and the first members are perfect squares;

$$\therefore (m^2A + m'^2B + m''^2C)G = (mAh + m'Bk + m''Cl)^2,$$

$$(n^2A + n'^2B + n''^2C)G = (nAh + n'Bk + n''Cl)^2,$$

$$(r^2A + r'^2B + r''^2C)G = (rAh + r'Bk + r''Cl)^2.$$

Hence, adding together these equations, and taking account of the equations of condition to which the quantities m, n, &c. are subject,

$$(A + B + C)G = A^2h^2 + B^2k^2 + C^2l^2,$$

or, restoring the value of G, and reducing,

$$A(B + C)h^2 + B(A + C)k^2 + C(A + B)l^2 = (A + B + C)D,$$

the equation to the locus, which is therefore a surface of the second order concentric with the proposed one, for there are only even powers of h, k and l.

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Cor. For the ellipsoid, replacing h, k, l by x, y, z, this equation becomes

$$(b^2+c^2)x^2+(a^2+c^2)y^2+(a^2+b^2)z^2=a^2b^2+a^2c^2+b^2c^2,$$

$$\text{or}\quad \frac{x^2}{g^2}+\frac{y^2}{g'^2}+\frac{z^2}{g''^2}=1,$$

if g, g', g'' denote the three altitudes of the triangle formed by joining the vertices of the ellipsoid (Cor. Art. 82.).

If we remove the origin to the extremity of the semiaxis a, and then make a infinite, the equations become

$$\frac{y^2}{2p}+\frac{z^2}{2p'}-2x=0,\quad y^2+z^2=4(p+p')x+4pp';$$

if therefore the given surface be a paraboloid, the locus required is a paraboloid of revolution.

If the given surface be a hyperboloid of one sheet, making c^2 negative, the equation to the locus is

$$(b^2-c^2)x^2+(a^2-c^2)y^2+(a^2+b^2)z^2=a^2b^2-c^2(a^2+b^2);$$

the locus is therefore an ellipsoid when c, the imaginary semiaxis, is less than $\frac{ab}{\sqrt{a^2+b^2}}$, and of course less than either of the real semiaxes; a point, viz. the center, when $c=\frac{ab}{\sqrt{a^2+b^2}}$; imaginary, till $c=b$ the less of the real semiaxes; a hyperboloid of two sheets till $c=a$, when it becomes a hyperbolic cylinder; and a hyperboloid of one sheet for all future values of c, *i. e.* whenever the imaginary axis is greater than either of the real axes.

If the given surface be a hyperboloid of two sheets, making b^2 and c^2 both negative, the equation to the locus

$$(b^2+c^2)x^2+(c^2-a^2)y^2+(b^2-a^2)z^2=a^2(b^2+c^2)-b^2c^2,$$

which is imaginary, as long as a the real semiaxis is less

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than $\frac{bc}{\sqrt{b^2+c^2}}$, and of course less than either b or c; a point, when $a = \frac{bc}{\sqrt{b^2+c^2}}$; afterwards an ellipsoid, till $a = c$ the less of the imaginary semiaxes, when it becomes an elliptic cylinder; a hyperboloid of one sheet, till $a = b$ when it becomes a hyperbolic cylinder; and for all future values of a, a hyperboloid of two sheets, *i. e.* whenever the real axis exceeds each of the imaginary axes.

Also, if in the original equation we put $D = 0$, $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$, we have the equation to a conical surface of the second order, and the equation to the locus becomes

$$(b^2 + c^2)\, x^2 + (a^2 + c^2)\, y^2 + (a^2 + b^2)\, z^2 = 0,$$

which is also the equation to a conical surface of the second order.

PROB. 7. To find the equation to the section of an oblique cone on a circular base.

Let the diameter of the base on which a perpendicular from the vertex falls, be taken for the axis of x. Then (Art. 49.) making $b = 0$, the equation to the surface is, putting $a - 2r = a'$,

$$c^2 y^2 = (cx - az)\, \{2cr + a'z - cx\}\,;$$

therefore, the equation to the section made by a plane through the axis of y inclined at an $\angle\theta$ to the plane of the base, is

$$c^2 y'^2 = (c\cos\theta - a\sin\theta)\, \{2crx' + (a'\sin\theta - c\cos\theta)\, x'^2\}.$$

As long as $\tan\theta$ lies between $\frac{c}{a}$ and $\frac{c}{a'}$, or θ between SAB and SEB (fig. 34.), *i. e.* as long as the plane cuts both sheets of the cone, the section is a hyperbola.

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When $\tan\theta = \frac{c}{a'}$; or the plane is parallel to ES, the section is a parabola. In other cases it is an ellipse, which becomes a circle when

$$c^2 = (c\cos\theta - a\sin\theta)\ (c\cos\theta - a'\sin\theta),$$

or, $$c^2 + c^2(\tan\theta)^2 = c^2 - c(a + a')\tan\theta + aa'(\tan\theta)^2;$$

$$\therefore\ \tan\theta = 0,\quad \text{or}\quad \tan\theta = \frac{c(a + a')}{aa' - c^2}.$$

The first value of θ gives the base of the cone; the second gives

$$\tan\theta = \tan(SAB + SEB),$$

or, if DAy be the plane of the circular section,

$$DAE = SAB + SEB,\quad \therefore\ DAS = BES;$$

that is, the cutting plane makes the same angle with one side of the principal section of the cone, that the base does with the other. This circular section is called the subcontrary section.

PROB. 8. The sum of the squares of the reciprocals of any three semidiameters of an ellipsoid mutually at right angles, is equal to the sum of the squares of the reciprocals of the semiaxes.

Let α, β, γ be the angles which any semidiameter r make with the axes, then the co-ordinates of its extremity are $x = r\cos\alpha$, $y = r\cos\beta$, $z = r\cos\gamma$;

$$\therefore\ \frac{1}{r^2} = \left(\frac{\cos\alpha}{a}\right)^2 + \left(\frac{\cos\beta}{b}\right)^2 + \left(\frac{\cos\gamma}{c}\right)^2;$$

and forming similar equations for r' and r'', and adding, since the semidiameters are mutually at right angles, we find

$$\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{r''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

PROB. 9. The area of a section of an ellipsoid made by a plane through its center, such that a perpendicular to it makes angles α, β, γ with the axes of the ellipsoid,

$$= \frac{\pi abc}{\sqrt{a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2}}.$$

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SECTION IV.

ON TANGENT PLANES AND NORMALS TO CURVE SURFACES, AND THEIR VOLUMES AND AREAS.

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107. To find the equation to the plane which touches a given curve surface at a proposed point.

Let $z = f(x, y)$ be the equation to the surface, and x, y, z the co-ordinates of the proposed point P (fig. 35.); then the equation to a plane passing through P will be

$$z' - z = A(x' - x) + B(y' - y),$$

x', y', z' denoting the co-ordinates of any point in it.

Through P draw a plane parallel to zx, cutting the surface in the curve PC, and the plane in the line PT; then these must touch one another at P, in order that the plane may be the tangent plane to the surface at that point. Now, for every point in the line PT, $y' = y$, and therefore its equation deduced from the equation to the plane is

$$z' - z = A(x' - x);$$

also the equation to the curve PC is deduced from $z = f(x, y)$ by making y constant in it; and in order that PT may touch PC, we must have $A = d_x z$, y being supposed constant in the differentiation. Again, through P draw a plane parallel to yz, cutting the surface in the curve PD and

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the plane in the line PR; then, as before, the equation to PR is

$$z' - z = B(y' - y);$$

and the equation to PD is obtained from $z = f(x, y)$ by regarding x as constant; and in order that PR may touch PD, we must have $B = d_y z$, x being supposed constant in the differentiation. Hence, substituting for A and B these values, the equation to the tangent plane at a point x, y, z, is

$$z' - z = d_x z\ (x' - x) + d_y z\ (y' - y);$$

or, as it is usually written,

$$z' - z = p\,(x' - x) + q\,(y' - y);$$

where p and q denote the partial differential coefficients of z derived from the equation to the surface, and the co-ordinates may be either rectangular or oblique.

Cor. The plane whose position is thus determined by the conditions that sections of it and the surface, made by planes parallel to two of the co-ordinate planes, touch one another, contains, as we shall hereafter shew, the tangent lines of all curves that can be drawn on the surface through the point in question. If the given equation to the surface, instead of having the above explicit form, should be $u = F(x, y, z) = 0$, then to determine p and q we have

$$d_{(x)}u + d_{(z)}u \,.\, p = 0, \quad d_{(y)}u + d_{(z)}u \,.\, q = 0,$$

the differential coefficients being formed as if x, y, z were independent; hence, substituting for p and q their values, the equation to the tangent plane under its most general form is

$$(x' - x)\, d_{(x)}u + (y' - y)\, d_{(y)}u + (z' - z)\, d_{(z)}u = 0.$$

108. If γ denote the angle of inclination of the tangent plane to that of xy, then (Art. 29.)

$$\cos\gamma = \frac{1}{\sqrt{1 + (d_x z)^2 + (d_y z)^2}};$$

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and similarly, its inclinations to the other co-ordinate planes may be determined.

109. To find the equations to the normal to a curve surface at a proposed point.

The normal is a straight line drawn through the point of contact perpendicular to the tangent plane; and will therefore have equations of the form

$$x' - x = m(z' - z), \quad y' - y = n(z' - z),$$

x, y, z being the co-ordinates of the point of contact; and since it is perpendicular to the plane whose equation is

$$z' - z = p(x' - x) + q(y' - y),$$

$$\therefore \text{ (Art. 25.)} \quad m + p = 0, \quad n + q = 0.$$

Hence, substituting for m and n their values, the equations to the normal at a point xyz, are

$$x' - x + p(z' - z) = 0, \quad y' - y + q(z' - z) = 0.$$

Cor. The angles α, β, γ formed by the normal with the axes of x, y, z produced in the positive directions, will be given (Art. 28.) by the formulæ

$$\cos\alpha = \frac{-p}{\sqrt{1 + p^2 + q^2}}, \quad \cos\beta = \frac{-q}{\sqrt{1 + p^2 + q^2}},$$

$$\cos\gamma = \frac{1}{\sqrt{1 + p^2 + q^2}};$$

or, if the equation to the surface be given under the form $u = F(x, y, z) = 0$, substituting for p and q their values (Cor. Art. 107.), we shall have

$$\cos\alpha = \frac{d_{(x)}u}{R}, \quad \cos\beta = \frac{d_{(y)}u}{R}, \quad \cos\gamma = \frac{d_{(z)}u}{R},$$

$$\text{where} \quad R = \sqrt{(d_{(x)}u)^2 + (d_{(y)}u)^2 + (d_{(z)}u)^2}.$$

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110. To find the length of the portion of the normal intercepted between the surface and the plane of xy, and the co-ordinates of the point where it meets that plane.

The co-ordinates of any point in the normal being x', y', z', and x, y, z those of the point where it meets the surface, the distance of these points is

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2} = (z'-z)\sqrt{1+p^2+q^2}.$$

But at the point where the normal meets the plane of xy, $z'=0$; $\therefore$ the required length $= -z\sqrt{1+p^2+q^2}$.

Also by making $z'=0$ in the equations to the normal, we find

$$x' = x + pz, \quad y' = y + qz,$$

the co-ordinates of the point where it meets the plane of xy; and similarly the points where the normal meets the other co-ordinate planes may be determined.

111. The equations to the normal may also be found from the consideration that it is the longest or shortest line which can be drawn from any point in itself to the surface.

Let x', y', z' be co-ordinates of a fixed point in the normal, and x, y, z of any point in the surface; then if δ = distance of these points,

$$\delta^2 = (x'-x)^2+(y'-y)^2+(z'-z)^2,$$

a function of two variables x and y, for z is given in terms of x and y, by the equation to the surface. Therefore, in order that δ^2 may be a maximum or minimum,

$$x'-x+d_x z(z'-z)=0, \quad y'-y+d_y z(z'-z)=0.$$

These are the equations which determine the position of the longest or shortest line that can be drawn from the fixed point $x'y'z'$, to the surface, and consequently those to the normal.

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112. To find the differential coefficient of the volume of a solid bounded by the co-ordinate planes and planes parallel to them, and by a given curve surface.

Let $z = f(x, y)$ be the equation to the surface, and x, y, z the co-ordinates of any point M (fig. 38.) in it. Through M draw planes parallel to the co-ordinate planes; then the volume V of the figure $NMEF$ is a function of x and y; and if x and y receive increments h and k, the point M will be brought to C, and the increment of the solid, contained by the planes ME, SD, SB, FM, will be

$$d_x V h + d_y V k + d_x^2 V \frac{h^2}{1.2} + d_x d_y V hk + d_y^2 V \frac{k^2}{1.2} + \&c.$$

But if we had made x alone vary, the increment would have been

$$MBRF = d_x V h + d_x^2 V \frac{h^2}{1.2} + \&c.;$$

similarly, if we had made y alone vary, the increment would have been

$$MDEQ = d_y V k + d_y^2 V \frac{k^2}{1.2} + \&c.;$$

therefore, subtracting, we have

$$\text{vol. } MCRQ = d_x d_y V hk + \&c.,$$

$$\text{or } \frac{\text{vol. } MCRQ}{hk} = d_x d_y V + \&c.;$$

hence, taking the limit of both sides by making h and k vanish, in which case vol. $MCRQ$, becoming ultimately a prism whose base is QR and altitude MP, $= zhk$, and all the terms in the second member after the first, since they involve different combinations of positive powers of h and k, disappear, we have

$$d_x d_y V = z.$$

113. Again, to find the differential coefficient of the surface, if S = area of the surface NM, by proceeding as above we may shew that

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surface $MC = d_x d_y S h k$ + &c.,

and therefore limit of $\frac{\text{surf. } MC}{hk} = d_x d_y S.$

But surf. MC is ultimately coincident with the portion of the tangent plane at M intercepted by the prism erected on the base QR, and therefore $= hk \sec \gamma$, if γ denote the inclination of the tangent plane at M to the plane of xy;

$$\therefore\ d_x d_y S = \sec \gamma = \sqrt{1 + (d_x z)^2 + (d_y z)^2}. \quad \text{(Art. 108.)}$$

114. In applying these results to find volumes and areas of curve surfaces by double integration, considerable attention is necessary in taking the integral between the proper limits. This will be best seen by an example. Suppose it were required to find the volume contained by two cylindrical surfaces (erected on given bases ON, AN' (fig. 37.) in the plane of xy whose equations are $x = \phi(y)$, $x = \psi(y)$,) and the curve surface BPP' whose equation is $z = f(x, y)$. Integrating first with respect to x, considering y as constant, the expression $d_x d_y V = z = f(x, y)$, between the limits

$$x = GN = \phi(y), \quad x = GN' = \psi(y),$$

we obtain a quantity, a function of y only, which when multiplied by δy is the ultimate value of the slice $PP'M$ contained between two planes whose distances from zx are y and $y + \delta y$ respectively; and if this be now integrated between the limits $y = b$, $y = b'$, we shall have the volume contained by two planes parallel to zx at distances therefrom respectively equal to b and b'.

Similarly, for the area of the surface, by integrating the expression

$$d_x d_y S = \sqrt{1 + (d_x z)^2 + (d_y z)^2} = F(x, y)$$

with respect to x, considering y as constant, between the limits $x = GN$, $x = GN'$, we obtain a quantity, a function of y only, which when multiplied by δy is the ultimate value of the portion of the surface PQ' contained between

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two planes parallel to zx at a distance δy from one another; and this integrated between the limits $y = b$, $y = b'$, gives us the area of the surface intercepted by any two planes parallel to zx.

115. To find the differential coefficients of the volume and surface of a solid, bounded by two of the co-ordinate planes and a plane parallel to the third, and by a plane through one of the co-ordinate axes and a given curve surface.

Let DMQ (fig. 40.) be a curve surface, QD being its trace in the plane of zx. Draw planes MP, Np parallel to yz, and MA, NA planes passing through Az. Let x, y, z be co-ordinates of the point M and $t = \tan \angle HAP$, therefore $y = xt$; also let $x + h$, $(x + h)(t + e)$ be co-ordinates of the point K. Then $\Delta AHP = \frac{1}{2}x^2 t$,

$$\text{and area } KH = (x+h)^2 \frac{e}{2} - x^2 \frac{e}{2} = he\left(x + \frac{h}{2}\right).$$

Now the volume DMP may be considered as a function of x and t; and, as before, by subtracting the increments arising from the separate variations of x and t from that arising from their simultaneous variations, it may be shewn that

$$\frac{\text{vol. } MKmk}{he} = d_x d_t V + \&c.;$$

hence, taking the limit of both sides, by making h and e vanish, in which case vol. $MKmk$, becoming ultimately a prism whose base is HK and altitude MH, $= zhe\left(x + \frac{h}{2}\right)$, we have

$$d_x d_t V = xz.$$

116. Again, if S = area of the surface DMQ, by a similar process we may shew that

$$\frac{\text{surf. } MN}{he} = d_x d_t S + \&c.;$$

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and taking the limit of both sides, in which case

$$\text{surf. } MN = HK \sec\gamma = he\left(x + \frac{h}{2}\right)\sec\gamma,$$

we have $$d_x d_t S = x \sec\gamma = x\sqrt{1 + (d_x z)^2 + (d_y z)^2}.$$

117. In applying these formulæ to examples, the substitution of xt for y gives

$$d_x d_t V = xz = xf(x, xt), \quad d_x d_t S = xF(x, xt);$$

integrating with respect to x, considering t as constant, between the limits $x = GN = \phi(t)$, $x = G'N' = \psi(t)$ (these values being obtained from the equations to the bounding cylinders AN, $A'N'$) (fig. 42.) we obtain quantities, functions of t only, which when multiplied by δt are respectively the ultimate values of the volume and surface of the wedge PM'; and these integrated between the limits $t = m$, $t = m'$, will give the volume and surface contained between two planes inclined to zx at angles whose tangents are respectively equal to m and m'.

118. In some cases it is convenient to express the differential coefficient of the volume of a solid by the polar co-ordinates explained in Art. 98.

Let ACP (fig. 39.) be a curve surface; AP, $A'P'$, its intersections with two conical surfaces described about OC with semivertical angles θ and $\theta + \delta\theta$; CP, CP' its intersections with two planes through CO inclined to zx at angles ϕ and $\phi + \delta\phi$; then it may be shewn as before, if $V =$ vol. $ACPO$, that

$$\frac{\text{pyramid } OPP'}{\delta\theta\,\delta\phi} = d_\theta d_\phi V + \&c.;$$

therefore, taking the limit of both sides by making $\delta\phi$, $\delta\theta$, vanish, in which case we may consider the base of the pyramid as coincident with the surface of a sphere, center O and radius OP, and therefore the area of the base will

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$= Pp \,.\, P'p = r \sin\theta \delta\phi \,.\, r\delta\theta$, and the vol. of the pyramid will $= \frac{r}{3} \,.\, r^2 \sin\theta \,\delta\phi \,\delta\theta$, we have

$$d_\theta d_\phi V = \frac{1}{3} r^3 \sin\theta.$$

119. If in the above expression we substitute for r its value in terms of ϕ and θ, and integrate with respect to θ, considering ϕ as constant, between the limits $\theta = COG = F(\phi)$, $\theta = COG' = F_1(\phi)$ (these values being obtained from the given equations to the bounding conical surfaces BOD, $B'OD'$) we shall obtain a quantity, a function of ϕ only, which when multiplied by $\delta\phi$ is the ultimate value of the volume of the wedge GOH'; and this, integrated between any two values of ϕ will give the volume OBD' contained between two planes inclined at those angles to zx, and between the given conical surfaces and the curve surface. If instead of the vertex O, we suppose the figure to be bounded by another curve surface whose equation is $r' = f_1(\phi, \theta)$, then the equation to be integrated will be

$$d_\theta d_\phi V = \frac{1}{3} (r^3 - r'^3) \sin\theta.$$

120. The following Examples will illustrate the method of drawing tangent planes and normals to curve surfaces, and finding their volumes and areas.

PROB. 1. To find the equation to the tangent plane to an ellipsoid at a proposed point.

The equation to the surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\therefore \quad \frac{x}{a^2} + \frac{z}{c^2} d_x z = 0, \quad \frac{y}{b^2} + \frac{z}{c^2} d_y z = 0;$$

Hence, substituting for $d_x z$, $d_y z$, their values in

$$z' - z = d_x z \,(x' - x) + d_y z \,(y' - y),$$

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the equation to the plane, which touches the ellipsoid at the point xyz, is

$$z' - z + \frac{c^2}{a^2}\frac{x}{z}(x' - x) + \frac{c^2}{b^2}\frac{y}{z}(y' - y) = 0,$$

$$\text{or} \quad \frac{zz'}{c^2} - \frac{z^2}{c^2} + \frac{xx'}{a^2} - \frac{x^2}{a^2} + \frac{yy'}{b^2} - \frac{y^2}{b^2} = 0,$$

$$\text{or} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

Cor. To find where the tangent plane cuts the axes, making $y' = z' = 0$, we have $x' = \frac{a^2}{x}$, the distance from the center at which it cuts the axis of x; similarly, the distances from the center at which it cuts the axes of y and z are $\frac{b^2}{y}$ and $\frac{c^2}{z}$.

Prob. 2. To find the equations to the normal to an ellipsoid at a proposed point.

Since $d_x z = -\frac{c^2}{a^2}\frac{x}{z}$, $d_y z = -\frac{c^2}{b^2}\frac{y}{z}$, the equations are

$$x' - x = \frac{c^2}{a^2}\frac{x}{z}(z' - z), \quad y' - y = \frac{c^2}{b^2}\frac{y}{z}(z' - z).$$

Cor. By making $z' = 0$, we find the co-ordinates of the point where the normal meets the plane of xy,

$$x' = x\left(1 - \frac{c^2}{a^2}\right), \quad y' = y\left(1 - \frac{c^2}{b^2}\right);$$

also, the length of the normal, intercepted between the surface and the plane of xy, is

$$= z\sqrt{1 + \frac{c^4}{a^4}\frac{x^2}{z^2} + \frac{c^4}{b^4}\frac{y^2}{z^2}} = c^2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}.$$

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PROB. 3. To find the locus of the extremity of the perpendicular dropped from the center upon the tangent plane to an ellipsoid.

The equations to a perpendicular from the center on the tangent plane are

$$x' = \frac{c^2}{a^2}\frac{x}{z}z', \quad y' = \frac{c^2}{b^2}\frac{y}{z}z',$$

which, combined with the equation to the tangent plane, give by eliminating x and y,

$$\frac{z}{c^2}\left(\frac{x'^2}{z'} + \frac{y'^2}{z'} + z'\right) = 1;$$

$$\therefore\ z = \frac{c^2 z'}{x'^2 + y'^2 + z'^2}, \quad x = \frac{a^2 x'}{x'^2 + y'^2 + z'^2}, \quad y = \frac{b^2 y'}{x'^2 + y'^2 + z'^2};$$

in which expressions x', y', z' are the co-ordinates of the point of intersection of the tangent plane and perpendicular, and x, y, z the co-ordinates of the point in the surface to which the tangent plane is applied.

$$\text{But} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\therefore\ (ax')^2 + (by')^2 + (cz')^2 = (x'^2 + y'^2 + z'^2)^2,$$

the required equation.

PROB. 4. If three tangent planes to an ellipsoid are mutually at right angles, their point of intersection will trace out a sphere concentric with the ellipsoid.

Let d, the perpendicular from the center on a tangent plane, make angles α, β, γ with the axes of x, y, z; then the equation to that plane is

$$\frac{x'\cos\alpha}{d} + \frac{y'\cos\beta}{d} + \frac{z'\cos\gamma}{d} = 1,$$

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which must be identical with the equation to the tangent plane expressed by the co-ordinates of the point of contact, viz.

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1;$$

$$\therefore \cos\alpha = \frac{xd}{a^2}, \quad \cos\beta = \frac{yd}{b^2}, \quad \cos\gamma = \frac{zd}{c^2}.$$

Hence since $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,

$$d^2 = \left(\frac{xd}{a}\right)^2 + \left(\frac{yd}{b}\right)^2 + \left(\frac{zd}{c}\right)^2 = (a\cos\alpha)^2 + (b\cos\beta)^2 + (c\cos\gamma)^2,$$

which gives the length of the perpendicular in terms of its inclinations to the axes. Now let there be two other perpendiculars d', d'', which make with the axes the angles α', β', γ'; α'', β'', γ''; respectively,

$$\therefore d'^2 = (a\cos\alpha')^2 + (b\cos\beta')^2 + (c\cos\gamma')^2,$$

$$d''^2 = (a\cos\alpha'')^2 + (b\cos\beta'')^2 + (c\cos\gamma'')^2;$$

and suppose the three perpendiculars to be mutually at right angles; then since α, α', α'' are the angles which a line (viz. the axis of x) makes with three rectangular axes, viz. the three perpendiculars, if we take account of the condition to which they, as well as β, β', β'', γ, γ', γ'', are subject (Art. 5), and add, we find

$$d^2 + d'^2 + d''^2 = a^2 + b^2 + c^2;$$

but the first member is equal to the square of the distance from the origin of the point of intersection of the three planes; this point therefore describes a spherical surface concentric with the ellipsoid whose radius $= \sqrt{a^2 + b^2 + c^2}$.

In the case of hyperboloids, one at least of the quantities a^2, b^2, c^2 is negative, and hence their sum may be negative or nothing; in the former case there is no point

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in space through which three rectangular planes touching the hyperboloid can be drawn, and in the latter, the center is the only point which has that property.

Cor. If we remove the origin to the extremity of the semiaxis a, and then make a infinite as in Art. 72, the equation to the ellipsoid will become

$$\frac{y^2}{2p} + \frac{z^2}{2p'} = 2x,$$

and the equation to the sphere

$$x^2 - 2ax + y^2 + z^2 = 2ap - p^2 + 2ap' - p'^2,$$

or, making a infinite,

$$-x = p + p';$$

therefore the locus, in the case of paraboloids, is a plane perpendicular to the axis of the surface, and at a distance $= -(p + p')$ from the vertex.

Prob. 5. Three planes mutually at right angles constantly touch the perimeter of a plane curve of the second order; to find the locus of their point of intersection.

Let the equation to the curve, and to one of the rectangular planes, be respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$x' \cos\alpha + y' \cos\beta + z' \cos\gamma = d;$$

then making $z' = 0$, the equation to the trace of the plane on xy is

$$\frac{x' \cos\alpha}{d} + \frac{y' \cos\beta}{d} = 1,$$

which must be identical with the equation to the tangent of the curve, viz.

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

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$$\therefore \cos\alpha = \frac{xd}{a^2}, \quad \cos\beta = \frac{yd}{b^2};$$

$$\therefore d^2 = \left(\frac{xd}{a}\right)^2 + \left(\frac{yd}{b}\right)^2 = (a\cos\alpha)^2 + (b\cos\beta)^2.$$

Similarly,

$$d'^2 = (a\cos\alpha')^2 + (b\cos\beta')^2, \quad d''^2 = (a\cos\alpha'')^2 + (b\cos\beta'')^2.$$

Hence, adding, $d^2 + d'^2 + d''^2 = a^2 + b^2$, which shews that the locus is a sphere, concentric with the curve, whose radius $= \sqrt{a^2 + b^2}$.

Cor. If the curve be a hyperbola, the radius $= \sqrt{a^2 - b^2}$; therefore we must have the transverse greater than the conjugate axis, otherwise the problem is impossible; if $a = b$, as in the rectangular hyperbola, the locus is a point, viz. the center. If the curve be a parabola, it may be shewn, as in Prob. 4, that the locus is a plane perpendicular to the plane of the parabola passing through its directrix.

Prob. 6. If two concentric surfaces of the second order have the same foci for their principal sections, they will cut one another every where at right angles.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$

be the equations to the surfaces; then the equations to the tangent planes are

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1, \quad \frac{xx'}{a'^2} + \frac{yy'}{b'^2} + \frac{zz'}{c'^2} = 1;$$

and in order that these may be at right angles, we must have

$$\frac{x^2}{a^2a'^2} + \frac{y^2}{b^2b'^2} + \frac{z^2}{c^2c'^2} = 0;$$

(hence one of the surfaces must be a hyperboloid, for some of the quantities a^2, b^2, &c. must be negative;) and this

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equation gives the relation among the co-ordinates of the points in which the surfaces may intersect at right angles. But by subtracting the two equations, we find

$$\left(\frac{1}{a^2} - \frac{1}{a'^2}\right) x^2 + \left(\frac{1}{b^2} - \frac{1}{b'^2}\right) y^2 + \left(\frac{1}{c^2} - \frac{1}{c'^2}\right) z^2 = 0,$$

for the relation among the co-ordinates of the actual points of intersection, which must be identical with the former if the surfaces intersect every where at right angles. Hence, equating the ratios of corresponding coefficients,

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

$$\text{or } a^2 - b^2 = a'^2 - b'^2, \quad a^2 - c^2 = a'^2 - c'^2, \quad b^2 - c^2 = b'^2 - c'^2,$$

which three equations express that the principal sections have the same foci.

Cor. In like manner, if the surfaces have not a center, and we represent their equations by

$$\frac{x^2}{c^2} + \frac{y^2}{b^2} + \frac{2x}{a} = 1, \quad \frac{x^2}{c'^2} + \frac{y^2}{b'^2} + \frac{2x}{a'} = 1;$$

by eliminating x between these equations, and comparing it with the equation expressing that the tangent planes are at right angles, we find that the surfaces will intersect every where at right angles, provided

$$\frac{a^2 - c^2}{2a} = \frac{a'^2 - c'^2}{2a'}, \quad \frac{a^2 - b^2}{2a} = \frac{a'^2 - b'^2}{2a'};$$

that is, the foci of the principal sections must be co-incident.

Prob. 7. If the equation to a surface be

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a},$$

the sum of the portions of the co-ordinate axes, intercepted between the origin and tangent plane at any point, is constant.

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121. Application of the formulæ for the volumes and areas of surfaces to examples.

I. To find the volume and area of the surface of a sphere, (fig. 41). Here

$$d_x d_y V = z = \sqrt{a^2 - x^2 - y^2};$$

$$\therefore\ d_y V = \tfrac{1}{2} x \sqrt{a^2 - y^2 - x^2} + \tfrac{1}{2}(a^2 - y^2) \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} + C,$$

and integrating between the limits $x = 0$, $x = MQ = \sqrt{a^2 - y^2}$,

$$d_y V = \frac{\pi}{4}(a^2 - y^2);$$

$\therefore$ integrating between the limits $y = 0$, $y = CM$,

$$\text{volume } AMBQ = \frac{\pi}{4}(a^2 y - \tfrac{1}{3} y^3),$$

and, making $y = a$, vol. $ABCD = \dfrac{\pi}{4} \cdot \dfrac{2a^3}{3} = \dfrac{\pi a^3}{6}$;

$$\therefore \text{ vol. of whole sphere} = 8 \frac{\pi a^3}{6} = \frac{4\pi a^3}{3}.$$

Again,

$$d_x d_y S = \sqrt{1 + (d_x z)^2 + (d_y z)^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = \frac{a}{\sqrt{a^2 - y^2 - x^2}};$$

$$\therefore\ d_y S = a \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} + C;$$

and taking the integral between the limits $x = 0$, $x = \sqrt{a^2 - y^2}$,

$$d_y S = \frac{\pi a}{2};$$

$\therefore$ integrating between the limits $y = 0$, $y = CM$,

$$\text{area of surface } PBAQ = \frac{\pi a y}{2},$$

and making $y = a$, area of surf. $DAB = \dfrac{\pi a^2}{2}$;

$$\therefore \text{ area of surface of whole sphere} = 8 \frac{\pi a^2}{2} = 4\pi a^2.$$

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II. A sphere is pierced by a cylinder the diameter of whose base is equal to the radius of the sphere, and their surfaces are in contact; to find the volume and area of the surface of that part of the sphere which is intercepted by the cylinder.

Taking the center of the sphere for origin, and planes drawn through the axis of the cylinder and perpendicular to it for those of zx and xy, and using the formula of Art. 115, we have

$$d_x d_t V = xz = x\sqrt{a^2 - x^2 - y^2} = x\sqrt{a^2 - (1+t^2)x^2};$$

$$\therefore\ d_t V = C - \tfrac{1}{3}\frac{1}{1+t^2}\left\{a^2 - (1+t^2)x^2\right\}^{\frac{3}{2}},$$

and taking the integral between the limits

$$x = 0, \quad x = AN = a\cos^2 MAN = \frac{a}{1+t^2} \quad \text{(fig. 44.)},$$

$$d_t V = \frac{a^3}{3}\left\{\frac{1}{1+t^2} - \frac{t^3}{(1+t^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{a^3}{3}\left\{\frac{1}{1+t^2} - \frac{t}{(1+t^2)^{\frac{3}{2}}} + \frac{t}{(1+t^2)^{\frac{3}{2}}}\right\};$$

$$\therefore\ V = \frac{a^3}{3}\left\{\tan^{-1}t + \frac{1}{\sqrt{1+t^2}} - \tfrac{1}{3}\frac{1}{(1+t^2)^{\frac{3}{2}}}\right\} + C,$$

and integrating between the limits $t = 0$, $t = \infty$,

$$\text{vol. } ABPM = \frac{a^3}{3}\left\{\frac{\pi}{2} - \frac{2}{3}\right\} = \frac{\pi a^3}{6} - \frac{2a^3}{9}.$$

Now vol. $ABCD = \dfrac{\pi a^3}{6}$; therefore the part of $ABCD$ not comprized in the cylinder $= \dfrac{2a^3}{9}$. Consequently, if the sphere be pierced by two equal cylinders, the part not comprized in them $= \dfrac{2}{9}$ (diameter)^3.

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Again,

$$d_x d_t S = x\sqrt{1+(d_x z)^2+(d_y z)^2} = x\frac{a}{z} = \frac{ax}{\sqrt{a^2-x^2(1+t^2)}};$$

$$\therefore\ d_t S = C - \frac{a}{1+t^2}\sqrt{a^2-x^2(1+t^2)};$$

and taking the integral between the limits $x=0$, $x=\frac{a}{1+t^2}$,

$$d_t S = a^2\left\{\frac{1}{1+t^2} - \frac{t}{(1+t^2)^{\frac{3}{2}}}\right\};$$

$$\therefore\ S = a^2\left(\tan^{-1}t + \frac{1}{\sqrt{1+t^2}}\right) + C,$$

and taking the integral between the limits $t=0$, $t=\infty$,

$$\text{area of surf. } BDP = a^2\left(\frac{\pi}{2}-1\right).$$

Now area of surf. $BCD = \frac{\pi a^2}{2}$; therefore part of BCD not intercepted $= a^2$; and if the whole sphere be penetrated by two equal cylinders, the area of the surface not comprized in them $= 8a^2 = 2\,(\text{diameter})^2$.

III. The axes of two equal cylinders intersect at right angles; to find the volume and superficial area of the portion which is common to both.

Let OC (fig. 46.) be the axis of one cylinder, OA that of the other; then the equation to AB, or to the cylinder whose base it forms, is $z = \sqrt{a^2-y^2}$;

$$\therefore\ d_x d_y V = \sqrt{a^2-y^2},$$

$$d_y V = x\sqrt{a^2-y^2} + C,$$

and integrating between the limits $x=0$, $x = ON = \sqrt{a^2-y^2}$,

$$d_y V = a^2 - y^2;$$

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$\therefore$ integrating again between the limits $y = 0$, $y = a$,

$$\text{vol. } ADMB = \frac{2a^3}{3},$$

$$\text{and whole volume intercepted} = \frac{16a^3}{3}.$$

Again, $d_y x = \dfrac{-y}{\sqrt{a^2 - y^2}}$, $d_z x = 0$;

$$\therefore\ d_z d_y S = \frac{a}{\sqrt{a^2 - y^2}};$$

$$\therefore\ d_y S = \frac{ax}{\sqrt{a^2 - y^2}} + C = a,$$

between the limits $x = 0$, $x = \sqrt{a^2 - y^2}$;

$\therefore$ area of surf. $ADPB = a^2 =$ area of $CDPB$;

$\therefore$ area of whole surface $= 16a^2$.

IV. To find the volume of the figure generated by a straight line which moves parallel to the plane of yz, and constantly passes through the axis of x and through the perimeter of a circle whose plane is perpendicular to the axis of y, and center in that axis.

If $b =$ distance of center of the circle from the origin, and $a =$ its radius,

$$\text{vol. generated} = \frac{\pi a^2 b}{2}.$$

V. To find the volume of a spherical sector which is inclosed by the spherical surface, and a conical surface having the center of the sphere for its vertex and one of the circles of the sphere for its base.

Let a be the radius of the sphere, and α the semi-vertical angle of the cone; then employing the formula of Art. 118,

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$$d_\theta d_\phi V = \frac{a^3}{3} \sin\theta;$$

$$\therefore\ d_\phi V = -\frac{a^3}{3}\cos\theta + C = \frac{a^3}{3}(1 - \cos\alpha)$$

between the limits $\theta = 0$, $\theta = \alpha$; therefore integrating again between the limits $\phi = 0$, $\phi = 2\pi$,

$$\text{volume of sector} = \frac{2\pi a^3}{3}(1 - \cos\alpha).$$

VI. To find the volume of any wedge of an ellipsoid intercepted between two planes passing through an axis, (fig. 41.)

$$d_x d_t V = xz = cx\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = cx\sqrt{1 - x^2\left(\frac{1}{a^2} + \frac{t^2}{b^2}\right)};$$

$$\therefore\ d_t V = C - \tfrac{1}{3}\frac{c}{\frac{1}{a^2} + \frac{t^2}{b^2}}\left\{1 - x^2\left(\frac{1}{a^2} + \frac{t^2}{b^2}\right)\right\}^{\frac{3}{2}};$$

and integrating between the limits $x = 0$,

$$x = MQ = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{t^2}{b^2}}},$$

$$\left(\text{for in the plane of } xy,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;\quad \therefore\ \frac{1}{a^2} + \frac{t^2}{b^2} = \frac{1}{x^2}\right),$$

we have

$$d_t V = \tfrac{1}{3}\frac{c}{\frac{1}{a^2} + \frac{t^2}{b^2}};$$

$$\therefore\ V = \frac{abc}{3}\tan^{-1}\left(\frac{at}{b}\right) = \text{vol. of wedge } ABQC,$$

and making t infinite, vol. $ABDC = \frac{abc}{3}\cdot\frac{\pi}{2}$;

$$\therefore\ \text{vol. of whole ellipsoid} = \frac{4\pi abc}{3}.$$

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VII. To find the area of the surface of an ellipsoid which differs little from a sphere. (Fig. 41.)

Let DR be a section parallel to the plane of xy at a distance $= c\cos\theta$ from it, and BR a section through the axis of z, inclined to the plane of zx at an $\angle\psi$, and let $\tan\psi = \frac{b}{a}\tan\phi$; and let S = area of surface BDR; then S is a function of θ and ϕ, and if $D'R'$, BR' be sections corresponding to the angles $\theta + \delta\theta$ and $\phi + \delta\phi$, we shall have

$$d_\phi d_\theta S + \&c = \frac{\text{area of surf. } RR'}{\delta\phi\,\delta\theta}.$$

Now if $OR = \rho$, $O'r = \rho + \delta\rho$, and γ = inclination of tangent plane at R to xy,

area RR' = (its projection on xy) $\sec\gamma = \rho\,\delta\rho\,\delta\psi\sec\gamma$, ultimately.

But the equation to RD is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = (\sin\theta)^2$;

$$\therefore \rho^2\left\{\left(\frac{\cos\psi}{a}\right)^2 + \left(\frac{\sin\psi}{b}\right)^2\right\} = (\sin\theta)^2,$$

$$\text{or, } \left(\frac{\rho\cos\psi}{a}\right)^2\{1 + (\tan\phi)^2\} = (\sin\theta)^2,$$

$$\therefore \rho = a\sin\theta\frac{\cos\phi}{\cos\psi}, \quad \text{and } \delta\rho = a\cos\theta\frac{\cos\phi}{\cos\psi}\delta\theta;$$

$$\text{also } \delta\psi = \frac{b}{a}\left(\frac{\cos\psi}{\cos\phi}\right)^2\delta\phi, \quad \text{ultimately;}$$

$$\therefore \text{area } RR' = ab\sin\theta\cos\theta\,\delta\phi\,\delta\theta\sec\gamma;$$

$$\therefore d_\phi d_\theta S = ab\sin\theta\cos\theta\frac{c}{z}\sqrt{1 - \frac{a^2-c^2}{a^2}\cdot\frac{x^2}{a^2} - \frac{b^2-c^2}{b^2}\cdot\frac{y^2}{b^2}},$$

$$= ab\sin\theta\sqrt{1 - \{\mu(\cos\phi)^2 + \nu(\sin\phi)^2\}(\sin\theta)^2},$$

$$\text{making } \frac{a^2-c^2}{a^2} = \mu, \quad \frac{b^2-c^2}{b^2} = \nu,$$

and observing that $x = \rho\cos\psi = a\sin\theta\cos\phi$, and $y = b\sin\theta\sin\phi$.

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Hence, making $\mu(\cos\phi)^2 + \nu(\sin\phi)^2 = u$ (which is a small quantity in the case proposed) and expanding,

$$d_\phi d_\theta S = ab\sin\theta\left(1 - \tfrac{1}{2}u(\sin\theta)^2 - \frac{1.1}{2.4}u^2(\sin\theta)^4 - \&c.\right)$$

$\therefore$ integrating between the limits $\theta = 0$, $\theta = \frac{\pi}{2}$, (Int. Cal. p. 116.)

$$d_\phi S = ab\left(1 - \tfrac{1}{3}u - \frac{1}{3.5}u^2 - \frac{1}{5.7}u^3 - \&c.\right)$$

and integrating again between the limits $\phi = 0$, $\phi = \frac{\pi}{2}$, we find the area of the surface of $\frac{1}{8}$th of the ellipsoid

$$= \frac{\pi ab}{2}\left\{1 - \tfrac{1}{3}P_1 - \frac{1}{3.5}P_2 - \frac{1}{5.7}P_3 - \&c.\right\},$$

where $P_1 = \frac{1}{2}(\mu + \nu)$, $P_2 = \mu^2\frac{1.3}{2.4} + \mu\nu\frac{1.1}{2.2} + \nu^2.\frac{1.3}{2.4}$;

&c. and in general P_n is the coefficient of x^n in the developement of $(1 - \mu x)^{-\frac{1}{2}}.(1 - \nu x)^{-\frac{1}{2}}$.

VIII. To find the area of the surface of an oblique cone on an elliptic base.

Let the perpendicular VM (fig. 47.) upon the base from the vertex of the cone fall upon CA the major axis of the ellipse, $CM = f$, $VM = h$; $x = a\sin\phi$, $y = b\cos\phi$, co-ordinates of P, $BP = s$, VPT the tangent plane along VP, cutting VMP' in VQ, VH a perpendicular upon PT, S = area of surf. BVP, then ultimately

$d_\phi S.\delta\phi$ = surf. VPP' = triangle $VPQ = \frac{1}{2}PQ.VH = \frac{1}{2}VH.\delta s$;

$\therefore\ d_\phi S = \frac{1}{2}VH\,d_\phi s = \frac{1}{2}\sqrt{h^2\{a^2 - c^2(\sin\phi)^2\} + b^2(a - f\sin\phi)^2}$,

as will be found upon calculating VH, since (Int. Cal. p. 159.)

$$d_\phi s = \sqrt{a^2 - c^2(\sin\phi)^2},$$

where $c^2 = a^2 - b^2$; this can only be integrated by elliptic functions, unless h and f be so related that $b^2f^2 = c^2(h^2 + b^2)$, when the quantity under the vinculum is a perfect square, and

$$\text{whole surface of cone} = \frac{\pi abf}{c}.$$

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SECTION V.

ON TANGENTS, AND NORMAL AND OSCULATING PLANES TO CURVES, AND THEIR LENGTHS.

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122. To show how a curve in space may be represented by equations.

We have seen (Art. 44.) that the equation $F(x, y, z) = 0$ represents a surface, and if to this we join another equation $F_1(x, y, z) = 0$ representing a second surface, and suppose the variables to receive only such values as satisfy both equations at the same time, we shall determine a series of points situated in each of the surfaces, that is, in the curve of their intersection. Conversely, as we have no other means of determining a curve in space than by assigning two surfaces each of which contains it, we cannot represent it analytically except by two simultaneous equations among the variables x, y, z. Among the various surfaces which may pass through a curve and so determine it, we employ for the sake of simplicity the *cylindrical* surfaces which are parallel to the co-ordinate axes, as their equations will contain only two of the variables. (Art. 43.)

Let QPR (fig. 43.) be a curve in space, and through all its points draw perpendiculars Pp, Rr, &c. to the plane of xy (or, if the axes be oblique, parallel to Az); the assemblage of these lines will form a cylindrical surface, called a *projecting cylinder* of QR, and meeting the plane of xy in the curve qpr, which is called the projection of QR on xy. Similarly, if we drop perpendiculars from all the points of QR on yz and zx, we shall have two other projecting cylinders, and two other projections; and the curve will evidently be determined if any two of

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its projections are given, for in that case two cylindrical surfaces will be given of which it is the intersection. Now the projections qpr, $Q'P'R'$ are determined by equations of the form

$$\phi_1(x, y) = 0 \quad \text{or} \quad y = \phi(x), \quad \psi_1(z, x) = 0 \quad \text{or} \quad z = \psi(x),$$

the complete signification of which, as we know, is the cylindrical surface erected upon each as its base. Therefore the curve QPR will be determined by the system of simultaneous equations

$$y = \phi(x), \quad z = \psi(x).$$

In these, only one of the variables, x for example, is arbitrary; and any assumed value of x joined to the corresponding values of y and z derived from them, will belong to a point in the curve. The equation to the projection of QPR on the plane of yz, is deduced from the other two by eliminating x.

Hence it appears that every curve in space may be considered as formed by the intersection of two cylindrical surfaces erected on its projections as their bases, perpendicular to the co-ordinate planes.

123. To find the equations to the line which touches a given curve at a proposed point.

Let PT (fig. 43.) be a tangent to the curve QPR at a point P; also let qpr be the projection of the curve on the plane of xy, and pt a tangent to the projection at p; we must first prove that pt is the projection of PT. Let R be a point in the curve near to P, r its projection; draw the lines PR, pr; then pr is the projection of PR, and continues so, however near R approaches to P, and consequently r (which is always in a line through R parallel to Pp) to p. Therefore in the ultimate positions of the lines, when R coincides with P and r with p, and they become tangents at P and p, pt is the projection of PT; that is, the projection of the tangent at any point coincides with the line touching the projection of the curve at the

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corresponding point. The same is of course true relative to the other co-ordinate planes; if therefore $y = \phi(x)$, $z = \psi(x)$ be the equations to the projections qr, $Q'R'$, the equations to the lines pt, $P'T'$, or to the line PT of which they are the projections, will be

$$y' - y = d_x y\,(x' - x), \quad z' - z = d_x z\,(x' - x);$$

x, y, z being the co-ordinates of the point of contact, and x', y', z' those of any point in the tangent, and the co-ordinates being either rectangular or oblique.

Cor. If the equations to the curve, instead of having the above explicit forms, should be

$$u = F(x,\ y,\ z) = 0, \quad u_1 = F_1(x,\ y,\ z) = 0,$$

then considering y and z each as a function of x, we have

$$d_{(x)}u + d_{(y)}u\ d_x y + d_{(z)}u\ d_x z = 0,$$

$$d_{(x)}u_1 + d_{(y)}u_1\, d_x y + d_{(z)}u_1\, d_x z = 0;$$

from which $d_x y$, $d_x z$ may be obtained and substituted in the above equations.

124. Hence we can shew that the tangent lines of all curves drawn on a surface through any given point in it, lie in the tangent plane at that point.

Let $z = f(x,\ y)$ be the equation to the surface, and p, q, the partial differential coefficients of z; also let $y = \phi(x)$ be the equation to the projection of any curve drawn upon it, then $z = f\{x, \phi(x)\}$ is the other equation to the curve; hence considering x, y, z as co-ordinates of a point in the curve, we have

$$d_x z = d_{(x)}f(x,\ y) + d_{(y)}f(x,\ y)\ d_x y = p + q\,d_x y;$$

$\therefore$ the equations to the line touching the curve at the point $x\,y\,z$, are

$$y' - y = d_x y\,(x' - x)$$

$$z' - z = (p + q\,d_x y)\,(x' - x);$$

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and these equations shew that, whatever be $d_x y$, the line coincides with the tangent plane to the surface at the point xyz, whose equation is

$$z' - z = p(x' - x) + q(y' - y).$$

The only case in which this theorem fails is at singular points when the partial differential coefficients p and q assume the form $\frac{0}{0}$; as at the vertex of a cone, or of a surface of revolution whose generating curve does not cut the axis at right angles.

Cor. Hence it follows that if a plane pass through a generating line of a cylinder or cone, and through a tangent to the base, it will be the tangent plane to the surface at every point in the generating line; for it will contain the tangents to all the curves traced on the surface through the various points of the generating line.

125. If the tangent to a curve makes angles α, β, γ with the axes of x, y, z, since its equations, taking the projections on the planes of zx and yz, are

$$x' - x = \frac{1}{d_z z}(z' - z), \quad y' - y = \frac{d_z y}{d_z z}(z' - z),$$

$$\cos\alpha = \frac{1}{\sqrt{1 + (d_z y)^2 + (d_z z)^2}}, \quad \cos\beta = \frac{d_z y}{\sqrt{1 + (d_z y)^2 + (d_z z)^2}},$$

$$\cos\gamma = \frac{d_z z}{\sqrt{1 + (d_z y)^2 + (d_z z)^2}}.$$

126. To find the equation to the normal plane to a curve at a proposed point.

A curve can have only one tangent at a proposed point, but it may have an infinite number of normals, that is, of lines perpendicular to the tangent through the point of contact; these all lie in one plane called the *normal plane*. Let x, y, z be the co-ordinates of the proposed point of

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the curve, then since the normal plane passes through that point, its equation will be

$$z' - z = A(x' - x) + B(y' - y);$$

and since it is perpendicular to the tangent whose equations are

$$x' - x = \frac{1}{d_z z}(z' - z), \quad y' - y = \frac{d_z y}{d_z z}(z' - z),$$

$$A + \frac{1}{d_z z} = 0, \quad B + \frac{d_z y}{d_z z} = 0;$$

hence, substituting for A and B these values, the equation to the normal plane at a point $x\ y\ z$, is

$$x' - x + (y' - y)\ d_z y + (z' - z)\ d_z z = 0.$$

127. To find the equation to the osculating plane to a curve at a proposed point.

The osculating plane at any point is that which has a closer contact with the curve than any other plane passing through the same point.

Let x, y, z be the co-ordinates of the point of the curve; then the equation to a plane passing through it will be

$$z' - z = A(x' - x) + B(y' - y);$$

let δ be the length of a perpendicular let fall on this plane from a contiguous point in the curve whose co-ordinates are $x + h$, $y + k$, $z + l$;

$$\therefore \delta = \frac{z + l - A(x + h) - B(y + k) - z + Ax + By}{\sqrt{1 + A^2 + B^2}} \quad \text{(Art. 37.)}$$

$$= \frac{l - Ah - Bk}{\sqrt{1 + A^2 + B^2}};$$

$$= \frac{h d_x z + \frac{1}{2} h^2 d_x^2 z + \&c. - Ah - B(h d_x y + \frac{1}{2} h^2 d_x^2 y + \&c.)}{\sqrt{1 + A^2 + B^2}},$$

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if we suppose the points consecutive, and substitute for k and l their developements in series ascending by powers of h. Now if we determine the constants A and B so that the coefficients of h and h^2 may vanish in the numerator of δ, the portion of the curve immediately contiguous to the point xyz will coincide more nearly with the plane so determined than with any other plane that can be drawn through that point.

$$\text{Let } \therefore d_x z - A - B d_x y = 0, \quad d_x^2 z - B d_x^2 y = 0;$$

$$\therefore B = \frac{d_x^2 z}{d_x^2 y}, \quad A = d_x z - \frac{d_x^2 z}{d_x^2 y} d_x y = \frac{d_x^2 y\, d_x z - d_x^2 z\, d_x y}{d_x^2 y},$$

$$\therefore z' - z = \frac{d_x^2 y\, d_x z - d_x^2 z\, d_x y}{d_x^2 y}(x' - x) + \frac{d_x^2 z}{d_x^2 y}(y' - y),$$

the equation to the osculating plane at a point xyz.

128. In order that the contact may be of the third order, the coefficient of h^3 in the numerator of δ must also vanish, or $d_x^3 z - B d_x^3 y = 0$;

$$\therefore d_x^2 z\, d_x^3 y - d_x^2 y\, d_x^3 z = 0,$$

the condition which must be satisfied at any point of the curve where the contact is of the third order. When the contact is of the second order,

$$\delta = \frac{1}{\sqrt{1 + A^2 + B^2}} \left\{ \frac{h^3}{6} (d_x^3 z - B d_x^3 y) + \&c. \right\},$$

a quantity which changes its sign at the same time that h does; consequently the curve generally cuts its osculating plane. On the contrary, when the contact is of the third order, the sign of δ does not change with that of h; and the curve is said in such cases to have a point of inflexion. The condition then of there being such a point is

$$+ \int d_x^2 z\, d_x^3 y - d_x^2 y\, d_x^3 z = 0.$$

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129. When a curve in space is a plane curve, if x, y, z be the co-ordinates of any point in it, they will always satisfy the equation to a plane

$$z' = Ax' + By' + c \ldots (1), \quad \text{so that } z = Ax + By + c;$$

therefore the differential coefficients of z and y will be such as to satisfy the equations

$$d_x z = A + B d_x y, \quad d_x^2 z = B d_x^2 y,$$

A and B denoting constant quantities;

$$\therefore d_x\left(\frac{d_x^2 z}{d_x^2 y}\right) = 0, \quad \text{or} \left(d_x^2 y\, d_x^3 z - d_x^2 z\, d_x^3 y = 0,\right)$$

which must be rendered identically true by the equations $y = \phi(x)$, $z = \psi(x)$, when they represent a plane curve; otherwise, the curve is of double curvature. This condition being satisfied, if we substitute for A, B, c their values in (1), we find the equation to the plane in which the curve is situated the same as that to the osculating plane, as might have been foreseen; for when a curve in space is a plane curve, the plane in which it is situated is the osculating plane at every point. Hence when the above condition is satisfied, the equation to the plane in which the curve is situated may be obtained by writing for the differential coefficients their values in the equation to the osculating plane.

Test of a plane curve.

130. The osculating plane at any point of a curve in space is perpendicular to the line of intersection of consecutive normal planes at that point.

The equation to the normal plane at a point xyz is

$$x' - x + (y' - y) d_x y + (z' - z) d_x z = 0;$$

and at the contiguous point $x+h$, $y + h d_x y +$ &c., $z + h d_x z +$ &c. it is

$$x' - x - h + (y' - y - h d_x y - \&c.)(d_x y + h d_x^2 y + \&c.)$$
$$+ (z' - z - h d_x z - \&c.)(d_x z + h d_x^2 z + \&c.) = 0;$$

or, combining it with the former and dividing by h, it is

$$-1 + (y' - y) d_x^2 y - (d_x y)^2 + (z' - z) d_x^2 z - (d_x z)^2 + \text{terms in } h, h^2, \&c. = 0.$$

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Therefore, making $h = 0$, the equations to the line of intersection of *consecutive* normal planes are

$$x' - x + (y' - y)\,d_x y + (z' - z)\,d_x z = 0,$$

$$(y' - y)\,d_x^2 y + (z' - z)\,d_x^2 z - (d_x y)^2 - (d_x z)^2 - 1 = 0;$$

and if we deduce the equations to the projections of this line on the planes of yz and zx, it will be seen that the conditions of being perpendicular to the osculating plane (Art. 25.) are fulfilled.

131. To find the differential coefficient of the length of the arc of a curve in space.

Let $y = \phi(x)$, $z = \psi(x)$ be the equations to the curve, x, y, z the co-ordinates of any point P in it (fig. 43.), $x + h$, $y + k$, $z + l$, the co-ordinates of a contiguous point R, s = length of arc QP, Q being a given fixed point in the curve, and R = chord PR;

then $R^2 = h^2 + k^2 + l^2$

$$= h^2 + (h\,d_x y + \frac{h^2}{2} d_x^2 y + \&c.)^2 + (h\,d_x z + \frac{h^2}{2} d_x^2 z + \&c.)^2$$

$$= h^2 \{1 + (d_x y)^2 + (d_x z)^2\} + A h^3 + \&c.;$$

$$\therefore\ d_x s = \text{limit of } \frac{R}{h} = \sqrt{1 + (d_x y)^2 + (d_x z)^2}.$$

Cor. Hence the formulæ for the angles α, β, γ, which a tangent at the point xyz forms with the axes of x, y, z, may be written

$$\cos\alpha = \frac{1}{d_x s},\quad \cos\beta = \frac{d_x y}{d_x s},\quad \cos\gamma = \frac{d_x z}{d_x s};$$

or, supposing s to be the independent variable,

$$\cos\alpha = d_s x,\quad \cos\beta = d_s y,\quad \cos\gamma = d_s z.$$

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132. The following are examples of finding the equations to curves in space, and drawing tangents, normal planes, &c.

1. A sphere is pierced by a cylinder the diameter of whose base is equal to the radius of the sphere, and their surfaces are in contact; to find the nature of the curve formed by their intersection. Take the center of the sphere for the origin, and planes drawn through the axis of the cylinder and perpendicular to it, for those of zx and xy respectively; and let $AN = x$, $NM = y$, $MP = z$ (fig. 44.) be the co-ordinates of a point P in the curve, and $AB = 2a$. Then because P is in the spherical surface, $x^2 + y^2 + z^2 = 4a^2$, and because M is a point in the semicircle $x^2 + y^2 = 2ax$; therefore the equations to the projections on xy and zx, or the equations to the curve, are

$$y^2 = 2ax - x^2, \quad z^2 = 4a^2 - 2ax \ldots (1);$$

the latter representing a parabola BD, vertex B, and axis AB; if we eliminate x, we find $4a^2y^2 = 4a^2z^2 - z^4$ for the equation to the third projection on yz.

Differentiating equations (1), we find

$$d_x y = \frac{a - x}{y}, \qquad d_x^2 y = -\frac{a^2}{y^3};$$

$$d_x z = -\frac{a}{z}, \qquad d_x^2 z = -\frac{a^2}{z^3}.$$

Hence the equations to the tangent PT at a point xyz, are

$$y' - y = \frac{a - x}{y}(x' - x), \quad z' - z = -\frac{a}{z}(x' - x);$$

and the co-ordinates of its trace T on the plane of xy, making $z' = 0$, are

$$x' = 4a - x, \quad y' = \frac{y^3}{x^2}.$$

The equation to the normal plane is

$$x' - x + \frac{a - x}{y}(y' - y) - \frac{a}{z}(z' - z) = 0;$$

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and the equation to the osculating plane

$$(y'-y)y^3-(z'-z)z^3=(x'-x)\{az^2+(a-x)y^2\}.$$

Also if s = length of arc DP,

$$d_z s=\sqrt{1+(d_z y)^2+(d_z z)^2}=\sqrt{\frac{a}{2x}\cdot\frac{2a+x}{2a-x}};$$

$$\text{let } x=2a(\cos\theta)^2;$$

$$\therefore\ d_\theta s=-2a\sqrt{2}\sqrt{1-\tfrac{1}{2}(\sin\theta)^2};$$

therefore, integrating by elliptic functions between the limits $\theta=\frac{\pi}{2}$, $\theta=0$,

arc DPB = elliptic quadrant semiaxes $2a\sqrt{2}$, and $2a$.

2. To find the equations to the Helix.

Whilst the rectangle $ABCM$ (fig. 45.) revolves uniformly about its side AB, the point P moves uniformly along the parallel side MC, and generates a curve called a helix.

When the rectangle is in the plane of zx, let P be at M, and let the velocity of $P=n$ times the velocity of M, $\therefore$ $PM=n$. arc DM; also let $AN=x$, $NM=y$, $MP=z$ be co-ordinates of P, and $AM=a$,

$$\therefore\ z=na\cos^{-1}\frac{x}{a},\quad \text{and } y=\sqrt{a^2-x^2};$$

which are the required equations; the former may also be written $z=na\sin^{-1}\frac{y}{a}$.

Cor. Since the corresponding increments of DM and PM are always in a given ratio, the curve DP always cuts the generating line CM of the cylinder on which it is traced at the same angle; and the line touching DP is always inclined to the plane of xy at a constant angle α whose tangent $=n$; therefore also the length of the helix

$$DP=\sec\alpha\ .\ \text{arc } DM=a\sec\alpha\cos^{-1}\frac{x}{a}.$$

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3. Let A (fig. 36.) be the pole of the great circle xBy of a sphere whose center is C; and as the quadrant AB revolves uniformly about AC, let the point P move uniformly along it; to find the equations to the locus of P.

When the plane ACB coincides with zx, let P be at A, and let P's velocity $= \frac{1}{n}$ of that of the point B,

$$\therefore \text{ arc } Bx = n \,.\, \text{arc } AP;$$

also let $CN = x$, $NM = y$, $MP = z$, be co-ordinates of the point P, and $AC = a$,

$$\therefore \tan^{-1}\frac{y}{x} = n\cos^{-1}\frac{z}{a}, \quad \text{and } x^2 + y^2 + z^2 = a^2;$$

which are the required equations. If $n = 1$, these equations may be transformed into

$$ax = z\sqrt{a^2 - z^2}, \quad ay = a^2 - z^2.$$

Cor. To find the area of the spherical surface included between the co-ordinate planes and the path of P.

Let AB' be a position of the quadrant very near to AB, meeting the curve in P', and a section of the surface through P perpendicular to AC in p; then if $ACP = \theta$, $ACP' = \theta + \delta\theta$, we have the increment of the area $AxBP = PP'BB' = PpB'B$ ultimately; hence (Int. Cal. p. 156.)

$$\delta S = BB' \,.\, PM = n \,.\, P'p \,.\, PM = n \,.\, a\delta\theta \,.\, a\cos\theta, \text{ ultimately,}$$

$$\text{or } d_\theta S = na^2\cos\theta.$$

$$\text{Therefore area of surface } AxBP = na^2\sin\theta.$$

This result may of course be obtained by means of the formula of Art. 116.

Also if V be the volume contained between the co-ordinate planes and a conical surface whose directrix is the path of P, we have, making $\angle BCx = \phi$,

$$d_\phi d_\theta V = \tfrac{1}{3}a^3\sin\theta;$$

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$$\therefore\ d_\phi V = C - \frac{a^3}{3}\cos\theta = \frac{a^3}{3}\cos\frac{\phi}{n},\ \text{from}\ \theta = ACP = \frac{\phi}{n},\ \text{to}\ \theta = 90^0;$$

$$\therefore\ V = \frac{na^3}{3}\sin\frac{\phi}{n},\ \text{from}\ \phi = 0\ \text{to}\ \phi = BCx,$$

$$\text{or volume}\ PBxAC = \frac{na^3}{3}\sin\theta.$$

4. To determine the curve to be traced on the surface of a sphere so that its length shall always be equal to n times the ordinate z of the describing point.

Let x, y, z be the rectangular co-ordinates of the describing point, and r, ϕ, θ the polar co-ordinates, then (Art. 98.) $z = r\cos\theta$, $y = r\sin\theta\sin\phi$, $x = r\sin\theta\cos\phi$; and we are required to assign a relation between ϕ and θ which will give $s = nr\cos\theta$, or $(d_\theta s)^2 = n^2r^2(\sin\theta)^2$, since r is constant;

$$\therefore\ (d_\theta x)^2 + (d_\theta y)^2 + (d_\theta z)^2 = n^2r^2(\sin\theta)^2,$$

$$\text{or}\ \{\cos\theta\cos\phi - \sin\theta\sin\phi\, d_\theta\phi\}^2 + \{\cos\theta\sin\phi + \sin\theta\cos\phi\, d_\theta\phi\}^2$$

$$+ (\sin\theta)^2 = n^2(\sin\theta)^2,$$

$$\text{or}\ 1 + (\sin\theta)^2(d_\theta\phi)^2 = n^2(\sin\theta)^2;$$

$$\therefore\ d_\theta\phi = \frac{1}{\sin\theta}\sqrt{n^2(\sin\theta)^2 - 1};$$

therefore, integrating (Integ. Cal. p. 109.),

$$\phi + C = n\cos^{-1}\frac{n\cos\theta}{\sqrt{n^2-1}} + \sin^{-1}\frac{\cot\theta}{\sqrt{n^2-1}},$$

$$\text{or}\ \sin^{-1}\frac{y}{\sqrt{r^2-z^2}} + C = n\cos^{-1}\frac{nz}{r\sqrt{n^2-1}} + \sin^{-1}\frac{z}{\sqrt{n^2-1}\sqrt{r^2-z^2}},$$

the equation to the projection on the plane of yz; the constant C may be determined if any point be given through which the curve is to pass.

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5. To find the line of greatest inclination at any point of a curve surface; that is, among all the straight lines which touch the surface, to determine that whose inclination to the plane of xy is the greatest.

As all these lines must lie in the tangent plane at the proposed point (Art. 124.), the line required is obviously that which is perpendicular to the trace of the tangent plane on xy. Now the equation to that trace is

$$y' - y = -\frac{p}{q}(x' - x) - \frac{z}{q};$$

$$\therefore\ y' - y = \frac{q}{p}(x' - x)$$

is the equation to a line perpendicular to it, passing through the point (x, y), and is therefore the equation to the projection on the plane of xy of the line required; the other equation

$$x' - x = \frac{p}{p^2 + q^2}(z' - z)$$

is obtained by combining the former with the equation to the tangent plane (in which the line in question lies) so as to eliminate $y' - y$.

6. Hence we can determine the curve of greatest inclination on any surface, that is, one whose tangent is always inclined at the greatest angle to the plane of xy.

Suppose $y = \phi(x)$ the equation to the projection of the curve, then $y' - y = d_x y\,(x' - x)$ is the equation to the line touching the projection at a point (x, y), which must be identical with the equation to the projection of the line of greatest inclination,

$$\therefore\ d_x y = \frac{q}{p}.$$

If, therefore, we determine p and q from the equation to the surface, independent of z, and substitute them in the

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above equation and integrate, the result will represent a curve on which if a cylinder be erected, it will intersect the surface in a curve having the property that its tangent is always inclined at the greatest angle to the plane of xy.

Ex. Let the surface be of the second order represented by the equation

$$Ax^2 + By^2 + Cz^2 = D, \quad \therefore\ p = -\frac{Ax}{Cz}, \quad q = -\frac{By}{Cz};$$

$$\therefore\ d_x y = \frac{By}{Ax}; \quad \text{or, integrating,} \quad y^A = C' x^B.$$

Hence, the curves of greatest inclination are of double curvature, and are projected horizontally into parabolic curves if A and B have the same sign; if, however, the point of departure, which determines the constant C', be in a principal plane, zx for instance, we have $y = 0$ when $x = x'$, and therefore $C' = 0$; hence the curve of greatest inclination is the principal section in zx, since its equation is reduced to $y = 0$.

7. On a given surface to determine the curve of constant inclination, that is, one whose tangent is always inclined at a constant angle to the plane of xy.

Let $z = f(x, y)$ be the equation to the surface, p and q the partial differential coefficients of z; also suppose $y = \phi(x)$ to be the equation to the projection of the curve of constant inclination, and γ the angle at which the tangent at a point xyz is inclined to the axis of z; then (Art. 131.)

$$\cos\gamma = d_s z = \frac{d_x z}{d_x s},$$

$$\therefore\ (d_x s)^2 = \{1 + (\tan\gamma)^2\}\ (d_x x)^2,$$

or,
$$1 + (d_x y)^2 = (\tan\gamma)^2\ (d_x x)^2;$$

but, x, y, z being co-ordinates of a point in the curve, we have $z = f(x, y)$ with the condition $y = \phi(x)$;

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$$d_x z = d_{(x)} f(x, y) + d_{(y)} f(x, y)\, d_x y = p + q d_x y,$$

$$\therefore \tan\gamma\,(p + q d_x y) = \sqrt{1 + (d_x y)^2};$$

and it remains to substitute for p and q their values in terms of x and y, and to integrate this equation. We shall then have the required relation $y = \phi(x)$.

8. When the given surface is one of revolution, the equation to the projection of the curve of constant inclination on a plane perpendicular to the axis, may be conveniently expressed by polar co-ordinates ϕ, ρ, so that

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

As above, we have

$$1 + (d_x y)^2 = (\tan\gamma)^2 (d_x z)^2,$$

$$\therefore (d_\rho x)^2 + (d_\rho y)^2 = (\tan\gamma)^2 (d_\rho z)^2,$$

or
$$1 + \rho^2 (d_\rho \phi)^2 = (\tan\gamma)^2 \cdot (d_\rho z)^2;$$

which may also be put under the form

$$(d_\rho z)^2 = (\cot\gamma)^2 \cdot \frac{\rho^2}{\rho^2 - p^2},$$

p being the perpendicular from the origin upon the tangent to the required projection. Now from the equation to the generating curve $z = f(\rho)$, we can find $d_\rho z$ in terms of ρ, and hence obtain the differential equation to the required projection.

Ex. Let the surface be a paraboloid, and the origin in the center of its base, c its altitude, a its semi-latus rectum;

$$\therefore (c - z)\, 2a = \rho^2, \quad d_\rho z = -\frac{\rho}{a}, \quad \therefore \frac{\rho^2}{a^2} = (\cot\gamma)^2 \cdot \frac{\rho^2}{\rho^2 - p^2},$$

$$\text{or, } \rho^2 - p^2 = (a\cot\gamma)^2,$$

the equation to the involute of a circle whose radius $= a\cot\gamma$.

If the surface be a right cone, the projection is an equiangular spiral.

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9. To determine a curve traced on a surface of revolution, which always cuts the generating curve at the same angle.

Let DP (fig. 48.) be the curve, EN its projection on a plane perpendicular to the axis of the surface; CPG a section of the surface through its axis meeting the curve in P, $PN = z$, $DP = s$, $ON = \rho$, $\angle AON = \phi$, and α the constant angle CPD at which the curve cuts the meridian; also let CQP', a meridian plane very near to CPG and inclined to zx at an angle $\phi + \delta\phi$, cut the curve in P', and a section of the surface through P perpendicular to its axis in Q; then

$$\sin CP'D = \frac{PQ}{PP'} = \frac{\rho\delta\phi}{\delta s} \text{ ultimately;}$$

or, taking the limit,

$$\sin\alpha = \frac{\rho}{d_\phi s}, \quad \therefore\ (d_\phi s)^2 = \rho^2 + \rho^2(\cot\alpha)^2,$$

$$\text{or } (d_\phi x)^2 + (d_\phi y)^2 + (d_\phi z)^2 = \rho^2 + (d_\phi \rho)^2 + (d_\phi z)^2 = \rho^2 + \rho^2(\cot\alpha)^2;$$

$$\therefore\ (d_\phi\rho)^2\{1 + (d_\rho z)^2\} = \rho^2(\cot\alpha)^2,$$

$$\text{or } d_\rho\phi = \frac{\tan\alpha}{\rho}\sqrt{1 + (d_\rho z)^2},$$

by integrating which (since $z = f(\rho)$ by the equation to the generating curve), we obtain the equation to the projection.

Ex. Let the surface be an oblate spheroid;

$$\therefore\ z = \frac{c}{a}\sqrt{a^2 - \rho^2}, \quad d_\rho z = -\frac{c\rho}{a\sqrt{a^2 - \rho^2}},$$

$$\therefore\ d_\rho\phi = \frac{\tan\alpha}{\rho}\sqrt{\frac{a^2 - e^2\rho^2}{a^2 - \rho^2}}, \quad \text{where } e^2 = \frac{a^2 - c^2}{a^2};$$

to integrate this, make $\dfrac{a^2 - e^2\rho^2}{a^2 - \rho^2} = v^2$, and the result is

$$\phi + C = \tfrac{1}{2}\tan\alpha\left\{\log_\epsilon\left(\frac{v-1}{v+1}\right) - e\log_\epsilon\left(\frac{v-e}{v+e}\right)\right\}.$$

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If the surface be a right cone, the projection of the curve is an equiangular spiral; if a sphere radius a, the equation between the radius vector and perpendicular on the tangent to the projection, is

$$\frac{1}{p^2} = \left(\frac{\operatorname{cosec} a}{\rho}\right)^2 - \left(\frac{\cot a}{a}\right)^2.$$

10. To determine the shortest line that can be drawn between two given points on a surface of revolution.

Let OC (fig. 48.) be the axis of the surface which take for the axis of z, and a plane AOB perpendicular to it for that of xy; DP a portion of the shortest line $= s$, x, y, z the co-ordinates of P, $ON = \rho$, $\angle AON = \phi$,

then $(d_\rho s)^2 = (d_\rho x)^2 + (d_\rho y)^2 + (d_\rho z)^2 = 1 + \rho^2 (d_\rho \phi)^2 + \{f'(\rho)\}^2$,

if $z = f(\rho)$ be the equation to the generating curve, and $d_\rho z = f'(\rho)$;

$$\therefore\ s = \int_\rho \sqrt{1 + x^2 p^2 + \{f'(x)\}^2} = \int_\rho V,$$

putting $\rho = x$, $\phi = y$, $d_\rho \phi = p$, in order to adapt the expression to the usual formulæ of the Calculus of Variations. Hence

$$N = d_{(y)} V = 0, \quad P = d_{(p)} V = \frac{x^2 p}{\sqrt{1 + x^2 p^2 + \{f'(x)\}^2}};$$

but $N - d_x P + \&c. = 0$, for a maximum or minimum; therefore in this case $P =$ a constant,

$$\text{or} \quad \frac{x^2 p}{\sqrt{1 + x^2 p^2 + \{f'(x)\}^2}} = \rho^2 \frac{d_\rho \phi}{d_\rho s} = c;$$

$$\therefore\ \frac{\rho}{d_\phi s} = \frac{c}{\rho}, \quad \text{or } \sin CPD \propto \frac{1}{ON},$$

as shewn in Prob. 9.

Hence the shortest line always cuts the meridian at an angle whose sine varies inversely as the distance of the point of intersection from the axis; and the equation to

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its projection on a plane perpendicular to the axis of the surface is

$$\rho^4 (d_\rho \phi)^2 = c^2 (d_\rho s)^2 = c^2 \{1 + \rho^2 (d_\rho \phi)^2 + (d_\rho z)^2\},$$

$$\text{or } d_\rho \phi = \frac{c}{\rho} \sqrt{\frac{1 + (d_\rho z)^2}{\rho^2 - c^2}},$$

c being the distance from the axis at which the shortest line cuts the meridian perpendicularly. Also for the length of the shortest line we have $d_\rho s = \frac{\rho^2}{c} d_\rho \phi$.

Ex. 1. Let the surface be a right cone, and $mz = \rho\sqrt{1-m^2}$, the equation to the generating line, then

$$d_\rho \phi = \frac{c}{\rho} \frac{1}{m\sqrt{\rho^2 - c^2}};$$

$$\therefore \phi + \alpha = \frac{1}{m} \sec^{-1} \frac{\rho}{c}, \quad \text{or } \rho = c \sec m(\phi + \alpha),$$

the equation to the projection, which is therefore an elliptic spiral.

Ex. 2. Let the surface be a sphere, and let DP be the arc of a great circle, then

$$\sin P = \frac{\sin CD}{\sin CP} \sin D;$$

if now the meridian revolve round OC,

$$\sin P \propto \frac{1}{\sin CP} \propto \frac{1}{ON};$$

hence an arc of a great circle is the shortest line that can connect two points on the surface of a sphere; this result may be obtained from the formula.

Ex. 3. To determine the shortest line on the surface of an oblate spheroid.

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Let CEB (fig. 49.) be the plane of the equator, PE, PN two meridians, AB a portion of the shortest line cutting PE perpendicularly at A. Let the point M be determined by the angle θ, so that $CT = \rho = a \sin\theta$, $TM = b\cos\theta$; and let A be similarly determined by the angle α, so that its distance from the axis $= a\sin\alpha$;

$$\text{then } d_\rho s = \rho \sqrt{\frac{1 + (d_\rho z)^2}{\rho^2 - a^2(\sin\alpha)^2}},$$

$$\therefore d_\theta s = \rho \sqrt{\frac{(d_\theta\rho)^2 + (d_\theta z)^2}{\rho^2 - a^2(\sin\alpha)^2}} = \sin\theta \sqrt{\frac{(a\cos\theta)^2 + (b\sin\theta)^2}{(\sin\theta)^2 - (\sin\alpha)^2}}.$$

Now, since α is the least value that θ can assume, let $\cos\theta = \cos\alpha\cos\psi$,

$$\therefore \frac{\sin\theta}{\cos\alpha\sin\psi} \cdot d_\psi s = \frac{\sin\theta}{\cos\alpha\sin\psi} \sqrt{b^2 + (a^2 - b^2)(\cos\alpha\cos\psi)^2},$$

$$\text{or } d_\psi s = \sqrt{(a\cos\alpha)^2 + (b\sin\alpha)^2 - (a^2 - b^2)(\cos\alpha)^2(\sin\psi)^2}$$

$$= a' \sqrt{1 - c^2(\sin\psi)^2},$$

$\therefore s =$ arc of an ellipse, amplitude ψ, whose semiaxes are $a' = \sqrt{(a\cos\alpha)^2 + (b\sin\alpha)^2}$, b, and eccentricity $= c$. At the point B, $\theta = 90^0$, $\therefore \psi = 90^0$, and $AB =$ a quadrant of the ellipse; also

$$\sin PMA = \frac{a\sin\alpha}{\rho} = \sin\alpha$$

at the point B, $\therefore \angle ABE = 90^0 - \alpha$.

For the projection on the plane of the equator,

$$\text{since } \rho^2 d_\rho\phi = a\sin\alpha\, d_\rho s, \quad \therefore d_\psi\phi = \frac{\sin\alpha}{a(\sin\theta)^2} d_\psi s,$$

$$\text{or, } d_\psi\phi = \frac{a'}{a\sin\alpha} \frac{\sqrt{1 - c^2(\sin\psi)^2}}{1 + n(\sin\psi)^2}, \text{ where } n = (\cot\alpha)^2,$$

$$= \frac{a'}{a\sin\alpha} \left\{ \frac{1 + \frac{c^2}{n} - \frac{c^2}{n}\{1 + n(\sin\psi)^2\}}{\{1 + n(\sin\psi)^2\}\sqrt{1 - c^2(\sin\psi)^2}} \right\};$$

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$$\therefore\ \phi = \frac{a'}{a \sin \alpha}\left\{\left(1+\frac{c^2}{n}\right)\Pi_c(n,\psi) - \frac{c^2}{n}F_c(\psi)\right\},$$

employing the notation of elliptic functions (Integ. Calc. p. 182.)

Cor. If the eccentricity of the generating ellipse be very small, let $\frac{a^2-b^2}{b^2} = \epsilon$, and $(\cos\alpha\cos\psi)^2 = m$;

$$\therefore\ d_\psi\phi = \frac{\sin\alpha}{a}\frac{\sqrt{b^2+(a^2-b^2)(\cos\alpha\cos\psi)^2}}{1-(\cos\alpha\cos\psi)^2} = \frac{\sin\alpha}{1-m}\sqrt{\frac{1+m\epsilon}{1+\epsilon}}$$

$$= \frac{\sin\alpha}{1-m}\left(1-\frac{1-m}{2}\epsilon+\frac{(1-m)(3+m)}{8}\epsilon^2-\&c.\right),$$

$$= \frac{\sin\alpha}{1-(\cos\alpha\cos\psi)^2} - \frac{\sin\alpha}{2}\left\{\epsilon-\frac{3\epsilon^2}{4}-\frac{\epsilon^2(\cos\alpha)^2}{8}(\cos\psi)^2-\&c.\right\};$$

$$\therefore\ \phi = \tan^{-1}\left(\frac{\tan\psi}{\sin\alpha}\right) - \psi\frac{\sin\alpha}{2}\left\{\epsilon-\frac{3\epsilon^2}{4}\right\}$$

$$+\frac{\epsilon^2}{16}(\psi+\tfrac{1}{2}\sin 2\psi)\sin\alpha(\cos\alpha)^2+\&c.$$

The indefinite continuation of the curve will manifestly produce an infinite number of spires similar and equal, and contained between two parallels each at a distance $= b\cos\alpha$ from the equator.

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SECTION V.*

ON THE DISCUSSION OF THE GENERAL EQUATION OF THE SECOND ORDER.

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133. To find the position of the center of any surface.

The center of a surface is a point O (fig. 22.) such that any chord of the surface PP', drawn through it, is bisected in it. (It must be observed, however, that if PP' cut the surface in more points than two, it would be sufficient that these points combined in a certain order should be, two and two, equally distant from O). If the surface be referred to any three axes originating in O, and PM, $P'M'$ be the ordinates parallel to Oz of the extremities of a chord, we see from the equal triangles POM, $P'OM'$, that these ordinates are equal and of contrary signs; the same thing would be true for the other co-ordinates of P and P', as well as for every other chord passing through O. If therefore $f(x, y, z) = 0$ be the equation to the surface, and if it be satisfied by $x = a$, $y = b$, $z = c$, it must also be satisfied by $x = -a$, $y = -b$, $z = -c$; that is, it must be such as not to alter when the signs of the three variables are changed; and, conversely, if it have this property, the origin is the center of the surface. When $f(x, y, z) = 0$ is algebraic, it cannot have the above property unless the dimension of every term be even in an equation of an even degree, and odd in an equation of an odd degree; for in the former case the equation is not at all altered by replacing x, y, z by $-x, -y, -z$; and in the latter (in which the equation cannot have a constant term) the sign of every term will be altered, and therefore the whole equation unaltered. Hence, to find whether a proposed surface admits

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of a center, we must refer it to parallel axes, through a new origin having co-ordinates h, k, l, by putting

$$x = x' + h, \quad y = y' + k, \quad z = z' + l,$$

and equate to nothing the coefficients of all the terms which are of a dimension different (as far as regards odd and even) from the degree of the equation; if these conditions can all be satisfied by real and finite values of h, k, l, the surface has a center, and h, k, l are its co-ordinates; in the contrary case the surface has no center.

134. To find the co-ordinates of the center of a surface of the second order represented by the general equation of the second degree.

The general equation of the second degree is

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0.$$

Here we must make $x = x' + h$, $y = y' + k$, $z = z' + l$, and equate to nothing the coefficients of terms of odd dimensions, that is, those involving x', y', z'. The result of these substitutions is

$$f(x' + h, y' + k, z' + l),$$

which when expanded will consist of three parts, (1) terms of two dimensions which must be the same as those of $f(x', y', z')$ which are of two dimensions; (2) terms of one dimension which are

$$x'd_h f(h, k, l) + y'd_k f(h, k, l) + z'd_l f(h, k, l),$$

each of which must disappear; and (3) a constant term $f(h, k, l)$; therefore the result will be

$$ax'^2 + by'^2 + cz'^2 + 2a'y'z' + 2b'z'x' + 2c'x'y' + f(h, k, l) = 0,$$

with the conditions $d_h f(h, k, l) = 0$, &c., or

$$\left.\begin{aligned} ah + b'l + c'k + a'' &= 0 \\ bk + a'l + c'h + b'' &= 0 \\ cl + a'k + b'h + c'' &= 0 \end{aligned}\right\}, \quad (1),$$

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for determining h, k, l. Multiplying the two latter equations respectively by indeterminate coefficients t, u, and adding them to the former, we have

$$(a + tc' + ub')\ h + a'' + tb'' + uc'' = 0,$$

provided $\quad c' + tb + ua' = 0, \quad b' + ta' + uc = 0;$

these two latter equations give t and u, and then substituting in the first we find $h = \dfrac{N}{D}$, where

$$D = aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c',$$

$$N = a''(ac - a'^2) + b''(a'b' - cc') + c''(a'c' - bb');$$

$$\text{similarly, } k = \frac{N'}{D}, \quad l = \frac{N''}{D}.$$

Hence, provided D is not $= 0$, these values of h, k, l, which are always real, are finite; and the surface will have a single center of which they are the co-ordinates, and its equation reckoned from the center as origin will be, suppressing accents,

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + a''h + b''k + c''l + d = 0;$$

for, multiplying equations (1) respectively by h, k, l, and adding, we have

$$f(h,\ k,\ l) = a''h + b''k + c''l + d.$$

135. If the coefficients of the given equation be such that $D = 0$, and the three numerators are not all $= 0$, at the same time, then one at least of the co-ordinates of the center will be infinite, which signifies that the surface has no center. If at the same time that $D = 0$ the three numerators vanish, then the surface admits of an infinite number of centers; for in that case the three equations (1) are reduced to one, or to two really distinct equations, and may therefore be satisfied by an infinite number of values of x, y, z. If they are reduced to two, that is, if the

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values of h and k deduced from the two first for instance, satisfy the third whatever l be, then there will be an infinite number of centers situated in a straight line which is the locus of the two independent equations; the surface will therefore be a cylinder on an elliptic or hyperbolic base. If the three equations (1) are reduced to a single equation, that is, if the value of h deduced from the first, for instance, satisfies the other two whatever k and l be, there will be an infinite number of centers situated in a plane which is the locus of the single independent equation, and the proposed surface will be a system of two planes parallel and equidistant from that plane. In this latter case the proposed equation must be capable of being resolved into two rational factors of the first degree.

136. The locus of the middle points of a system of parallel chords of any proposed surface is called its *diametral surface*. This surface will have several sheets, if each of the chords has more than two points in common with the proposed surface; if, for instance, the proposed surface be of the n^{th} order, the points of intersection with its chords, real or imaginary, will be in number n, and their combination on the same indefinite line will form $\frac{1}{2}n(n-1)$ different chords, and as many middle points; and therefore the diametral surface, since it may be met by an indefinite line in $\frac{1}{2}n(n-1)$ points, will have an equation of the degree $\frac{1}{2}n(n-1)$. For surfaces of the second order where $n=2$, the diametral surfaces can only be planes.

When any surface admits of a diametral plane, if we make it the plane of xy, and take the axis of z parallel to the chords which it bisects, the equation to the surface, for every pair of values $x=a$, $y=b$, must furnish for z values which, taken two and two, are equal and of contrary signs; and therefore the equation, supposed algebraic, can only involve even powers of z; and, conversely, whenever an equation contains only even powers of one of the variables, z for instance, the plane of xy is a diametral plane, and is said to be *conjugate* to the chords parallel to the axis

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of z. Also if a diametral plane be perpendicular to the chords which it bisects, it is called a *principal* plane, and the chords *principal* chords. Moreover the intersection of any two diametral planes is called a diameter of the surface.

137. To find the equation to a diametral plane of a surface of the second order.

Let $x = mz$, $y = nz$, be the given equations to a line through the origin to which the proposed system of chords is parallel, and $f(x, y, z) = 0$, the general equation of the second degree, the equation to the surface. Let h, k, l, be the co-ordinates of the middle point of any chord, and let the surface be referred to axes parallel to the former passing through it; then the equation will become

$$f(x' + h,\ y' + k,\ z' + l) = 0,$$

and the equations to the chord itself will be $x' = mz'$, $y' = nz'$; therefore the values of z' belonging to the point of the surface where the chord meets it, are given by the equation

$$f(mz' + h,\ nz' + k,\ z' + l) = 0 \ldots\ldots (1),$$

which is of the form

$$Rz'^2 + Sz' + T = 0;$$

and since the values of z' are equal and of opposite signs, $S = 0$. But S is the coefficient of z' in equation (1);

$$\therefore\ m d_h f(h,\ k,\ l) + n d_k f(h,\ k,\ l) + d_l f(h,\ k,\ l) = 0,$$

$$\text{or } m(ah + b'l + c'k + a'') + n(bk + a'l + c'h + b'')$$

$$+ cl + a'k + b'h + c'' = 0 \ldots\ldots (2),$$

$$\text{or } (am + c'n + b')h + (bn + c'm + a')k + (c + b'm + a'n)l$$

$$+ a''m + b''n + c'' = 0,$$

the relation among the co-ordinates of the middle point of any chord, or the equation to a diametral plane.

Cor. As the coefficients of h, k, l are possible, there will be a diametral plane for all values of the constants m

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and n, unless the three coefficients should all become nothing at the same time, when it will be situated at an infinite distance. But the equations

$$am + c'n + b' = 0, \quad bn + c'm + a' = 0, \quad c + b'm + a'n = 0,$$

containing only two unknown quantities, have an equation of condition which is the same as $D = 0$ (Art. 134.); in this case therefore the surface has not a center. When the surface has a center, every diametral plane passes through it, or through the locus of the centers; for equation (2) is visibly satisfied by the co-ordinates of the center furnished by equations (1), (Art. 134). This also follows from the definition.

138. Any diametral plane of a surface having a center is parallel to the tangent plane applied at the extremity of the diameter to which it is conjugate.

Taking the center for origin, the equation to the surface will be

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + d = 0;$$

therefore the equation to the tangent plane at a point xyz is (Cor. Art. 107),

$$(ax + b'z + c'y)(x' - x) + (by + a'z + c'x)(y' - y) + (cz + a'y + b'x)(z' - z) = 0,$$

$$\text{or } (ax + b'z + c'y)x' + (by + a'z + c'x)y' + (cz + a'y + b'x)z' + d = 0;$$

and if it be applied at the extremity of the diameter whose equations are $x = mz$, $y = nz$, the equation becomes

$$(am + c'n + b')x' + (bn + c'm + a')y' + (c + b'm + a'n)z' + \frac{d}{z} = 0,$$

and therefore (Art. 18.) represents a plane parallel to the diametral plane which is conjugate to the diameter $x = mz$, $y = nz$, the equation to which is (putting $a'' = b'' = c'' = 0$, in the equation Art. 137.)

$$(am + c'n + b')x' + (bn + c'm + a')y' + (c + b'm + a'n)z' = 0.$$

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This result might have been foreseen, since the lines in which a diametral plane and a tangent plane at the extremity of the conjugate diameter, are cut by any plane through that diameter must, by the nature of lines of the second order, be parallel to one another.

139. We shall now proceed to the reduction of the general equation of the second degree

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+2a''x+2b''y+2c''z+d=0,$$

where we suppose the co-ordinates rectangular; for if they were oblique, by transforming them to rectangular co-ordinates we should obtain an equation of the same degree as the above (Art. 96.), and which could not therefore be more general than the one which we have assumed. We shall prove, as affirmed at Art. 53, that this equation, after being simplified as much as possible, will always assume one or other of the forms

$$Ax^2+By^2+Cz^2=D,$$

$$By^2+Cz^2=2A'x,$$

the co-ordinates being rectangular; and therefore can never represent any other surface than one of those discussed in Section 2.

140. Every surface of the second order has at least one diametral plane which is perpendicular to the chords bisected by it.

Let the equation to the surface be

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+2a''x+2b''y+2c''z+d=0,$$

and $x=mz$, $y=nz$, the equations to the line to which a system of chords is parallel; then the equation to the plane which bisects the chords is (Art. 137.)

$$(am+c'n+b')x+(bn+c'm+a')y+(c+b'm+a'n)z$$
$$+a''m+b''n+c''=0,$$

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and our object is to shew that real values can be assigned to m and n, such that this plane shall be perpendicular to the chords. The conditions for this are (Art. 25.)

$$m = \frac{am + c'n + b'}{c + b'm + a'n}, \quad n = \frac{bn + c'm + a'}{c + b'm + a'n},$$

from which, by eliminating one of the unknown quantities m or n, we shall obtain a cubic which will always give a real value for the other; and the direction of the system of principal chords will be determined. But we shall obtain a more symmetrical result by assuming for the unknown quantity

$$\left.\begin{array}{lll} c + b'm + a'n = s, & \text{or } s - c = b'm + a'n, \\ \text{then } am + c'n + b' = ms, & \text{or } m(s-a) = c'n + b' \\ bn + c'm + a' = ns, & \text{or } n(s-b) = c'm + a' \end{array}\right\} \ldots(1).$$

Hence, determining m and n from the two latter equations,

$$\left.\begin{array}{l} m\{(s-a)(s-b) - c'^2\} = b'(s-b) + a'c' \\ n\{(s-b)(s-a) - c'^2\} = a'(s-a) + b'c' \end{array}\right\} \ldots(2).$$

And substituting for m and n in the former, we have

$$(s-a)(s-b)(s-c) - a'^2(s-a) - b'^2(s-b) - c'^2(s-c) - 2a'b'c' = 0,$$

$$\text{or } s^3 - (a+b+c)s^2 + (ab+ac+bc-a'^2-b'^2-c'^2)s$$

$$- (abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c') = 0.$$

This equation, being of an odd degree, will always have one real root which substituted in (2) will give real values for m and n; and therefore in every surface of the second order there is at least one principal plane, or, which is the same thing, one system of principal chords. Also there cannot be more than three, unless the particular form of the equation should render any two of the equations (1) identical, in which case m and n would be indeterminate, and the number would be infinite.

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Cor. That the cubic has all its roots real, may be shewn by putting it under the form

$$(s-c)\{(s-a)(s-b)-c'^2\}-\{a'^2(s-a)+b'^2(s-b)+2a'b'c'\}=0,$$

and substituting for s, α and β the roots of $(s-a)(s-b)-c'^2=0$. The results of these substitutions, since $(\alpha-a)(\alpha-b)=c'^2$, $(\alpha-\beta)(b-\beta)=c'^2$, are

$$-\{a'\sqrt{\alpha-a}\pm b'\sqrt{\alpha-b}\}^2 \text{ and } +\{a'\sqrt{\alpha-\beta}\pm b'\sqrt{b-\beta}\}^2,$$

for α is greater than both a and b, and β less. Therefore there is one root greater than a, another between a and β, and a third less than β.

141. Every variety of surfaces of the second order referred to rectangular co-ordinates is comprehended, without exception, in the equation

$$Ax^2+By^2+Cz^2+2A'x+2B'y+2C'z+D=0.$$

Let the surface be represented by the general equation of the second degree $f(x, y, z)=0$, and let it be referred to three new rectangular axes Ox', Oy', Oz', by the substitutions of Art. 90, and let the transformed equation be

$$Ax'^2+By'^2+Cz'^2+2A'y'z'+2B'z'x'+2C'x'y'+2A''x'$$
$$+2B''y'+2C''z'+D=0;$$

also let the equations to a line parallel to a system of principal chords (the existence of which in every case is certain) referred to the new axes, be $x'=mz'$, $y'=nz'$, then m and n satisfy the conditions

$$Am+C'n+B'=m(C+B'm+A'n)$$
$$Bn+C'm+A'=n(C+B'm+A'n).$$

Suppose now one of the new axes, that of z' for instance, to be parallel to the direction of the principal chords; therefore $m=0$, $n=0$; consequently we must have $A'=0$, $B'=0$. Consequently, whenever one of the rectangular axes is parallel to the direction of a system of principal chords, the

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general equation is freed from two of the rectangles, and takes the form

$$Ax'^2+By'^2+Cz'^2+2C'x'y'+2A''x'+2B''y'+2C''z'+D=0....(2).$$

This equation comprehends all surfaces of the second order without exception, and it may be still further reduced if, without altering the axis of z', we turn the axes of x' and y' in their own plane through an angle ϕ, so as to make the term involving $x'y'$ disappear, by the substitutions (Art. 94.)

$$x' = x_1 \cos\phi - y_1 \sin\phi,$$

$$y' = x_1 \sin\phi + y_1 \cos\phi;$$

this gives $2\cos\phi \sin\phi (A-B) + 2C' \{(\cos\phi)^2 - (\sin\phi)^2\} = 0,$

$$\text{or} \quad \tan 2\phi = -\frac{2C'}{A-B},$$

which value, being real and always admissible even when $A=B$, shews that equation (2) may be reduced to the form (suppressing accents)

$$Ax^2+By^2+Cz^2+2A'x+2B'y+2C'z+D=0;$$

which (the co-ordinates being rectangular) comprehends all surfaces of the second order.

142. It is at this point of the reduction of the general equation of the second order, that the separation of the surfaces represented by it into two classes takes place.

1. If none of the squares of the variables are wanting in the equation

$$Ax^2+By^2+Cz^2+2A'x+2B'y+2C'z+D=0,$$

i. e. if A, B, C are all different from zero, by writing $x+h$, $y+k$, $z+l$ for x, y, z and determining h, k, l by the conditions

$$Ah+A'=0, \quad Bk+B'=0, \quad Cl+C'=0,$$

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(which will give finite values for h, k, l) the origin will be changed, but the directions of the axes unaltered, and the equation will be reduced to the form

$$Ax^2 + By^2 + Cz^2 = D \ldots (1.)$$

which represents the first class of surfaces of the second order.

2. If one only of the squares is wanting, A for instance, and the corresponding coefficient A' is different from zero, then the term involving x cannot be made to disappear, for that would give h infinite; but instead of that we may determine h so as to exterminate the constant term, and the equation will be reduced to the form

$$By^2 + Cz^2 = 2A'x \ldots (2.)$$

which represents the second class of surfaces of the second order. The figures and properties of these two classes of surfaces have already been discussed, (Art. 53.).

143. There are still two particular cases to be examined, belonging, as we shall see, to the two sorts of cylindrical surfaces of the second order. First, suppose that the coefficient of one of the squares, A for instance, vanishes, and at the same time the corresponding coefficient $A' = 0$, without either of the other squares disappearing, the equation is then

$$By^2 + Cz^2 + 2B'y + 2C'z + D = 0,$$

which, containing only two of the variables, represents a cylinder perpendicular to the plane of yz on an elliptic or hyperbolic base; and if the center of this curve be taken for the origin, the equation will be reduced to the form

$$By^2 + Cz^2 = D,$$

which is deducible from equation (1) by making $A = 0$.

Secondly, suppose only one of the squares to remain in the equation, z^2 for instance; then writing $z + l$ and $x + h$ for z and x, and determining h and l so as to exterminate the term containing z, and the constant term, the equation becomes

$$Cz^2 + 2A'x + 2B'y = 0;$$

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now, without altering z, turn the axes of x and y in their own plane through an angle ϕ, such that $A' \sin\phi = B' \cos\phi$, then the equation will be reduced to the form

$$Cz^2 = 2A'x,$$

which is deducible from equation (2) by putting $B = 0$.

It is unnecessary to examine the case where the three squares disappear, as no equation of the second order could ever by transformation of co-ordinates be reduced to that form.

144. Hence, we conclude, that all surfaces of the second order with all their varieties, are comprised in the two classes represented by the equations

$$Ax^2 + By^2 + Cz^2 = D,$$

$$By^2 + Cz^2 = 2A'x,$$

the co-ordinates being rectangular. In the first class, each of the co-ordinate planes is a principal plane; therefore there are three systems of principal chords, and consequently the three roots of the cubic in (Art. 140.) are all real. In the second class, only the planes of zx and xy are principal planes; therefore there are at least two systems of principal chords; the third system, determined with the others by the cubic equation just referred to, must therefore be also real, but the corresponding principal plane is situated at an infinite distance. Hence it results from this discussion, as well as from the form of the equation (as proved in Cor. Art. 140.), that the three roots of the cubic equation which determines the positions of the principal chords are always real.

145. By means of the preceding results, we are able to determine the species and form of the surface represented by any proposed equation of the second order, without recurring to the laborious process of transformation of co-ordinates. We must first ascertain whether the surface has a center by the method of Art. 134, and if it has, find the co-ordinates of the center h, k, l from the equations which

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result, by equating to nothing the three derived equations of the first order, x, y, z being replaced by h, k, l; then taking that point as origin, the equation is reduced to

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+a''h+b''k+c''l+d=0,$$

from which the species of the surface and the values of the real or imaginary axes can be readily determined, as will be seen in the next Art. In the succeeding Articles, the characteristics of the different sorts of surfaces that have not a center will be deduced.

Ex. To find the center of the surface represented by the equation

$$2x^2+5y^2+3z^2+2yz-4zx-2xy+2x+8y-6z+8=0.$$

Forming the three derived equations, and replacing x, y, z by h, k, l, we have

$$\left.\begin{aligned} 4h-4l-2k+2&=0\\ 10k+2l-2h+8&=0\\ 6l+2k-4h-6&=0\end{aligned}\right\} \text{ which give } \begin{cases} h=&1,\\ k=&-1,\\ l=&2.\end{cases}$$

Hence, the equation referred to the center becomes

$$2x^2+5y^2+3z^2+2yz-4zx-2xy=-1+4+6-8=11.$$

146. The equation

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy=d,$$

(d being a positive quantity) belongs to an ellipsoid, hyperboloid of one sheet, or hyperboloid of two sheets, according as the cubic equation

$$(s-a)(s-b)(s-c)-a'^2(s-a)-b'^2(s-b)-c'^2(s-c)-2a'b'c'=0,$$

gives for s three positive values, two, or one.

Let $x=mz$, $y=nz$, be the equations to a line through the origin, which will be a diameter of the surface, since the origin is the center, and let the length of the diameter

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be denoted by $2r$; then the chords parallel to it will be bisected by a plane, of which the equation is

$$(am + c'n + b')x + (bn + c'm + a')y + (c + b'm + a'n)z = 0;$$

and if this plane be perpendicular to the chords, or if $2r$ be a principal diameter,

$$\text{making } c + b'm + a'n = s,$$

$$\text{we must have } \left.\begin{aligned} am + c'n + b' &= ms \\ bn + c'm + a' &= ns, \end{aligned}\right\} \ldots (1.)$$

from which eliminating m and n, as in Art. 140, we find

$$(s-a)(s-b)(s-c) - a'^2(s-a) - b'^2(s-b) - c'^2(s-c) - 2a'b'c' = 0;$$

also multiplying the two latter equations by m and n respectively, and adding them to the former, we find

$$s(1 + m^2 + n^2) = am^2 + bn^2 + c + 2a'n + 2b'm + 2c'mn.$$

But if x, y, z be the co-ordinates of the extremity of the diameter,

$$r^2 = x^2 + y^2 + z^2 = (1 + m^2 + n^2)z^2;$$

also since x, y, z are co-ordinates of a point in the surface,

$$z^2(am^2 + bn^2 + c + 2a'n + 2b'm + 2c'mn) = d;$$

$$\therefore\ r^2 = \frac{z^2}{s} \cdot s(1 + m^2 + n^2) = \frac{z^2}{s} \cdot \frac{d}{z^2}, \quad \text{or } r^2 = \frac{d}{s}.$$

Now d is a positive quantity, therefore according as s has three positive values, two, or one, r has three real values, two or one, or the surface has three real principal diameters, two, or one, and is therefore an ellipsoid, hyperboloid of one sheet, or hyperboloid of two sheets.

147. Hence, when we have an equation of the second order reckoned from the center, and the constant term forms one member of the equation and is positive, if with the numerical coefficients we form the above cubic, and observe the signs of its terms, we shall determine the nature of

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the surface represented by the proposed equation. For since all the roots of the cubic are real, there will be three positive roots, two, or one, according as there are in it supposed complete, three changes of signs, two, or one. If the cubic offers no change of signs, but only continuations, its three roots are all negative, and the three axes of the surface are imaginary, and consequently the surface itself imaginary. Also, if upon referring the equation to the center as origin the constant term d disappears, the surface represented is a cone, since if we combine its equation with the equation to a plane through the origin $z = Ax + By$, the result will be of the form $y = x(p \pm \sqrt{q})$, indicating two straight lines through the origin, unless the radical be impossible for all values of A and B, in which case the proposed equation represents a point.

Cor. If we solve the cubic, we may then easily calculate the lengths of the axes, and fix the position of each by means of the values of m and n, corresponding to each value of s, given by equations (1).

Ex. $2x^2 + 5y^2 + 3z^2 + 2yz - 4zx - 2xy = 11.$

The cubic is $s^3 - 10s^2 + 7s + 38 = 0$, which has two positive roots and one negative root; therefore the surface is a hyperboloid of two sheets.

148. When the proposed equation

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0,$$

is such that

$$abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c' = 0,$$

the surface represented either has not a center, or it has a central line or plane, Art. (135.), and admits of four varieties, viz., paraboloids, parabolic cylinders, elliptic or hyperbolic cylinders, and a system of two parallel planes; for each one of which, certain conditions must be satisfied which are not all true for the others. Hence we can determine the distinctive characteristics of each of these surfaces.

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1. For paraboloids, since the co-ordinate planes cannot be all parallel to the axis of the surface, and since it is only such planes which intersect the surface in a parabola, (Art. 103.) one of the quantities

$$a'^2 - bc, \quad b'^2 - ac, \quad c'^2 - ab, \quad (1)$$

must be different from zero, and negative in the case of the elliptic, and positive in that of the hyperbolic paraboloid; also one at least of the co-ordinates of the center will be infinite, and therefore one of the equations for determining those co-ordinates imaginary.

2. For parabolic cylinders, since all sections are either straight lines or parabolas, each of the quantities (1) vanishes, and one of the equations for determining the co-ordinates of the center will be imaginary.

3. For elliptic or hyperbolic cylinders, the equations for determining the co-ordinates of the center are reduced to two distinct equations, and one at least of the quantities (1) is different from zero, and is negative in the former, and positive in the latter, case. Also, by taking sections parallel to the co-ordinate planes, it will be necessary further to examine whether the cylinder is wholly imaginary, or reduced to a line, or to a system of two planes not parallel.

4. For a system of two parallel planes, the equations for determining the co-ordinates of the center must be reduced to a single distinct equation, and the three quantities (1) must each vanish. It will be further necessary, by taking sections, to examine whether the two planes are confounded in one, or are imaginary.

Ex. $x^2 - 2y^2 - 3yz + 3zx + xy + 4z = 0.$

The equations for determining the co-ordinates of the center are

$$2x + 3z + y = 0,$$

$$-4y - 3z + x = 0,$$

$$-3y + 3x + 4 = 0;$$

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and since the last subtracted from the sum of the two former gives $4 = 0$, this impossible equation shews that the surface has not a center. Also, $c'^2 - ab = (\frac{1}{2})^2 + 2$ which is different from zero and positive; therefore the surface is a hyperbolic paraboloid.

149. To find the relations among the coefficients of the general equation of the second order when it represents surfaces of revolution.

In a surface of revolution if we take any number of sections perpendicular to the axis, (which will be circles) and in each draw any number of chords all parallel to the same straight line, then this system of chords of the surface will all have their middle points in a plane through the axis at right angles to them. Hence, in every surface of revolution, there exists an infinite number of principal chords all parallel to the same plane, and if we express this property analytically we shall find the required relations.

Let the surface, referred to rectangular co-ordinates, be represented by the general equation of the second degree

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+2a''x+2b''y+2c''z+d=0,$$

and let $z = mz$, $y = nx$ be the equations to a principal chord through the origin; then m and n will be determined by any two of the equations (Art. 140.)

$$\begin{aligned} am + c'n + b' &= ms, \\ bn + c'm + a' &= ns, \\ c + b'm + a'n &= s, \end{aligned}$$

s being a root of the cubic, found by eliminating m and n; or since $m = \frac{x}{z}$, $n = \frac{y}{z}$, x, y, z being co-ordinates of any point in the principal chord through the origin, that line may be represented by any two of the equations

$$\left.\begin{aligned} (a-s)x + c'y + b'z &= 0, \\ (b-s)y + c'x + a'z &= 0, \\ (c-s)z + b'x + a'y &= 0. \end{aligned}\right\} \quad (1.)$$

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But if the surface be one of revolution, one value at least of s must be such as to reduce these three equations to a single distinct equation which will represent a plane, all chords parallel to which will be principal chords.

Let s' denote this value, then, equating the ratios of corresponding coefficients, we have

$$\frac{a-s'}{c'} = \frac{c'}{b-s'} = \frac{b'}{a'},$$

$$\frac{a-s'}{b'} = \frac{c'}{a'} = \frac{b'}{c-s'};$$

from which result two equations of condition, and the value of s',

$$a - \frac{b'c'}{a'} = b - \frac{a'c'}{b'} = c - \frac{a'b'}{c'} = s'.$$

Cor. Introducing these relations into equations (1), they are reduced to the single equation

$$b'c'x + a'c'y + a'b'z = 0, \quad (2.)$$

which represents a plane, all chords parallel to which are principal chords. Also if we substitute for a, b, c the values furnished by these equations in the cubic (Art. 140.), it will take the form

$$(s-s')^2 \left\{ s - s' - \frac{b'c'}{a'} - \frac{a'c'}{b'} - \frac{a'b'}{c'} \right\} = 0.$$

Hence, when the above conditions are satisfied, the cubic for determining the directions of the principal chords has two roots equal to s', so that of the three systems of principal chords two may be drawn in any manner parallel to the plane represented by equation (2).

150. To find the equations to the axis of revolution.

The equation to a principal plane corresponding to the root $s=s'$, is (Art. 140.)

$$m(s'x + a'') + n(s'y + b'') + s'z + c'' = 0,$$

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where m and n, instead of being determinable from s' by means of two equations, are now only subject to one equation, viz.

$$\frac{m}{a'} + \frac{n}{b'} + \frac{1}{c'} = 0;$$

therefore eliminating n,

$$m\left\{(s'x + a'') - \frac{b'}{a'}(s'y + b'')\right\} = \frac{b'}{c'}(s'y + b'') - (s'z + c''),$$

where m remains arbitrary. Therefore to determine a line which shall be common to all these planes, we must equate each of the members to zero, which gives

$$a'(s'x + a'') = b'(s'y + b'') = c'(s'z + c'').$$

These consequently are the equations to the line in which all the diametral planes intersect, or to the axis of revolution of the surface; they may, by dividing by s' and substituting for it the values given in Art. (149.), be put under the form

$$a'\left(x + \frac{a'a''}{aa' - b'c'}\right) = b'\left(y + \frac{b'b''}{bb' - a'c'}\right) = c'\left(z + \frac{c'c''}{cc' - a'b'}\right).$$

Cor. In order that the axis may be at a finite distance, the value of s' must be different from zero, a condition to be added to those of the preceding Art. Although these results may be so modified as to embrace all particular cases, yet when a proposed equation wants several terms, it will generally be better to apply the direct investigation, to discover whether or no it represents a surface of revolution.

151. Three diametral planes are said to be *conjugate* to one another, when each bisects the chords which are parallel to the intersection of the two others; and the intersections in that case are called conjugate diameters. When a surface admits of three planes of this sort and they are taken for the co-ordinate planes, its equation supposed algebraic, can only contain even powers of the three variables x, y, z. Hence the principal planes of a surface of the second order having a center, are conjugate to one another, for the equation referred to them is

$$Ax^2 + By^2 + Cz^2 = D;$$

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and this, as we have seen, is the only *rectangular* system of co-ordinates which can give the equation of this form, or which can be conjugate to one another.

152. We shall now shew that there is an infinite number of systems of diametral planes oblique to one another which, taken for the co-ordinate planes, will give the equation of the above form, that is, free from the terms involving zx, xy, yz, and which are therefore conjugate to one another.

Let $x = mz$, $y = nz$ be the equations to any diameter Oz' (fig. 50.), then the equation to the diametral plane $x'Oy'$ conjugate to Oz', is (Art. 137),

$$md_x f(x, y, z) + nd_y f(x, y, z) + d_z f(x, y, z) = 0,$$

or,
$$mAx + nBy + Cz = 0.$$

Since the chords which $x'Oy'$ bisects are parallel to Oz', the planes which are conjugate to $x'Oy'$, must according to the definition pass through Oz', and they will cut the plane $x'Oy'$ in two lines Ox', Oy', such that if together with Oz' they be taken for a system of oblique axes, the equation to the surface will only contain even powers of the three variables, and therefore be of the form

$$A_1x'^2 + B_1y'^2 + C_1z'^2 = D_1.$$

But if we make $z' = 0$, the result $A_1x'^2 + B_1y'^2 = D_1$ is the equation to the section $x'Oy'$, and by its form shews that the axes Ox', Oy' are conjugate diameters of that section; hence it follows, that any diameter Oz', being proposed, if in the section made by the diametral plane which is conjugate to it we draw any two conjugate diameters Ox', Oy', at pleasure, we shall determine three planes $z'Ox'$, $x'Oy'$, $y'Oz'$ which are conjugate to one another; the number of such systems is therefore unlimited; and the equation to the surface when referred to a system of conjugate diameters as axes, may be put under the form

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

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a', b', c' being the distances, real or imaginary, from the center at which the surface cuts them.

Cor. If Ox' be in a principal plane, the diametral plane conjugate to Ox' will be perpendicular to that plane. Also if Ox', Oy', be axes of the section $x'Oy'$, they will be at right angles to one another, or if Ox' coincide with an axis of the surface, it will be perpendicular both to Ox' and Oy'; but in no case will three conjugate diameters be mutually at right angles, unless they coincide with the axes of the surface.

153. We have seen (Art. 138.) that any diametral plane is parallel to the tangent plane, applied at the extremity of the diameter to which it is conjugate. Hence, in a system of conjugate diameters, the tangent plane at the extremity of each is parallel to the plane of the two others. Also, if through the extremities of each of a system of conjugate diameters we draw planes parallel to the plane of the two others, we shall form a parallelopiped, which is said to be constructed upon the diameters, and which in the case of the ellipsoid will be circumscribed about the surface. We shall now shew that the known properties of conjugate diameters of curves of the second order may be extended to surfaces of the second order.

154. In a surface of the second order, that has a center, the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the axes; also the volume, and the sum of the squares of the faces, of the parallelopiped constructed on any system of conjugate diameters, are respectively equal to the volume, and the sum of the squares of the faces, of the rectangular parallelopiped constructed on the axes.

Let $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$ be the equation to a surface of the second order, referred to a system of conjugate diameters, the inclinations of which are $\angle yz = \lambda$, $\angle zx = \mu$, $\angle xy = \nu$; and let $x = mz$, $y = nz$, be the equations to any diameter;

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then the equation to the diametral plane conjugate to this diameter is

$$\frac{mx}{a'^2} + \frac{ny}{b'^2} + \frac{z}{c'^2} = 0,$$

and if it be perpendicular to the diameter, we have (Art. 86.)

$$mc'^2(1 + m\cos\mu + n\cos\lambda) = a'^2(m + n\cos\nu + \cos\mu),$$
$$nc'^2(1 + m\cos\mu + n\cos\lambda) = b'^2(n + m\cos\nu + \cos\lambda).$$

Let $r^2 = c'^2(1 + m\cos\mu + n\cos\lambda)$;

$$\therefore\ mr^2 = a'^2(m + n\cos\nu + \cos\mu),$$
$$nr^2 = b'^2(n + m\cos\nu + \cos\lambda);$$

$$\therefore\ r^2\left(\frac{1}{c'^2} + \frac{m^2}{a'^2} + \frac{n^2}{b'^2}\right) = 1 + m^2 + n^2 + 2mn\cos\nu + 2m\cos\mu + 2n\cos\lambda;$$

$\therefore$ (as in Art. 146.) r is the length of the semidiameter whose equations are $x = mz$, $y = nz$. Also if we determine m and n from the two latter equations, and substitute them in the former, the result is

$$(r^2 - a'^2)(r^2 - b'^2)(r^2 - c'^2) - a'^2b'^2(r^2 - c'^2)(\cos\nu)^2 - a'^2c'^2(r^2 - b'^2)(\cos\mu)^2$$
$$- b'^2c'^2(r^2 - a'^2)(\cos\lambda)^2 - 2a'^2b'^2c'^2\cos\mu\cos\nu\cos\lambda = 0,$$

or $r^6 - r^4(a'^2 + b'^2 + c'^2) + r^2\{(a'b'\sin\nu)^2 + (a'c'\sin\mu)^2 + (b'c'\sin\lambda)^2\}$
$$-(a'b'c')^2\{1 + 2\cos\lambda\cos\mu\cos\nu - (\cos\lambda)^2 - (\cos\mu)^2 - (\cos\nu)^2\} = 0.$$

But if a, b, c be the three semiaxes of the surface, then a^2, b^2, c^2 are the values of r^2 in the above equation; therefore, by the theory of equations, we have

$$a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2,$$
$$(a'b'\sin\nu)^2 + (a'c'\sin\mu)^2 + (b'c'\sin\lambda)^2 = (ab)^2 + (ac)^2 + (bc)^2 \ldots\ldots(2),$$
$$(a'b'c')^2\{1 + 2\cos\lambda\cos\mu\cos\nu$$
$$- (\cos\lambda)^2 - (\cos\mu)^2 - (\cos\nu)^2\} = (abc)^2 \ldots\ldots(3),$$

which are the three required results, the first member of the third being the well-known expression for the square of the volume of a parallelopiped in terms of its three edges, and their inclinations to one another.

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Cor. When one or two of the axes are imaginary, there will be an equal number of the conjugate diameters imaginary; and it will be necessary to change the signs of the squares of these axes and these diameters in the above equations.

155. The preceding results may also readily be arrived at by the following method.

Let Ox, Oy, Oz (fig. 29.) be the semiaxes of the surface a, b, c; Ox', Oy', Oz' any system of semiconjugate diameters a', b', c'; let the plane of $x'y'$ intersect that of xy in the $\frac{1}{2}$ diameter $Ox_1 = a_1$; also let $Oy_2 = b_2$ be the semidiameter of the curve x_1x', which is conjugate to Ox_1;

$$\therefore\ a_1^2 + b_2^2 = a'^2 + b'^2.$$

Let the plane $z'Oy_2$ intersect xy in the semidiameter $Oy_1 = b_1$, then this plane must contain Oz; for being conjugate to Ox_1 in a principal plane, it must be perpendicular to that plane; hence Ox_1, Oy_1, Oz form a system of semiconjugate diameters, and any two are semiconjugate diameters of the plane section in which they are situated;

$$\therefore\ b_1^2 + c^2 = b_2^2 + c'^2,$$

$$a^2 + b^2 = a_1^2 + b_1^2;$$

therefore, adding these equations to the former,

$$a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2.$$

Again, employing the same auxiliary systems of diameters as above, and denoting each parallelopiped by its three edges, we have

$$\text{vol.}\,(a',\ b',\ c') = \text{vol.}\,(a_1,\ b_2,\ c'),$$

for these figures have the same altitude, viz. the perpendicular from z' on the plane of $x'y'$, and equal bases, viz. the parallelograms constructed on the two systems of $\frac{1}{2}$ conjugate diameters a', b', a_1, b_2, belonging to the same curve in the plane of $x'y'$. Similarly,

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$$\text{vol.}\,(a_1,\ b_2,\ c') = \text{vol.}\,(a_1,\ b_1,\ c)$$

$$\text{vol.}\,(a_1,\ b_1,\ c) = \text{vol.}\,(a,\ b,\ c)\,;$$

$$\therefore\ \text{vol.}\,(a',\ b',\ c') = \text{vol.}\,(a,\ b,\ c).$$

Cor. If the conjugate diameters $2a'$, $2b'$, $2c'$ are each equal to $2R$, then $3R^2 = a^2 + b^2 + c^2$; also since there are only two equations, viz. (2) and (3) Art. 154, to determine the angles of inclination λ, μ, ν of the conjugate diameters, there may be an infinite number of systems of equal conjugate diameters, their extremities all lying in the intersection of the surface and a concentric sphere radius $= R$. And of all systems of conjugate diameters of an ellipsoid the principal diameters have their sum a minimum and the equal diameters their sum a maximum. For in any proposed system, if there were two not perpendicular to one another, by substituting for them the axes of the section of which they are diameters, we should obtain a system whose sum is less than that of the proposed system. Again, if there were two not equal to one another, by substituting for them the equal diameters of the section to which they belong, we should determine a system whose sum is greater than that of the proposed one. And in this manner we could shew that no oblique system could have their sum a minimum, and no unequal system their sum a maximum.

156. In surfaces of the second order not having a center, represented by the equation to rectangular co-ordinates

$$By^2 + Cz^2 = 2A'x,$$

the planes of zx and xy only are diametral and principal, each bisecting perpendicularly the chords parallel to the intersection of the other with the plane of yz, which is the tangent plane at the vertex. In this case no three diametral planes can be conjugate to one another; for, taking a system of chords parallel to a line whose equations are $x = mz$, $y = nz$, the equation to the diametral plane conjugate to them is

$$Bny + Cz = A'm \ldots\ldots (1).$$

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Therefore all the diametral planes are parallel to the axis of x, and consequently their intersections, which are the diameters of the surface, are parallel to that axis, and cannot therefore form a system of co-ordinate axes.

157. But we can find an infinite number of oblique co-ordinate planes related to one another in the same manner as the three rectangular planes mentioned above are, viz. so that two, $z'x'$ and $x'y'$, shall be diametral planes, and each of them conjugate to the chords parallel to the intersection of the other with $y'z'$; the latter being a tangent plane to the surface at the extremity of the diameter in which $z'x'$ and $x'y'$ intersect; and the equation to the surface when referred to them will therefore preserve the same form.

For let $x'O'y'$ (fig. 51.) be the diametral plane represented by equation (1), and which, being parallel to the axis of the surface, cuts it in a parabola $AO'B$. In $AO'B$ take any point O', and draw $O'z'$ parallel to the chords to which the diametral plane is conjugate. Then the two co-ordinate planes which are to go along with this diametral plane will pass through $O'z'$, and cut the diametral plane in two lines $O'x'$, $O'y'$, such that being taken for the co-ordinate axes they shall give the equation to the surface under the form

$$B_1 y'^2 + C_1 z'^2 = 2A_1' x'.$$

Therefore, making $z' = 0$, the equation to $AO'B$ is

$$B_1 y'^2 = 2A_1' x',$$

which by its form shews that it is referred to a diameter of the parabola and a tangent to the parabola at the extremity of that diameter. Hence the position of the second diametral plane $z'O'x'$, and also of the third co-ordinate plane $z'O'y'$, is determined, which latter, since it passes through two tangent lines to the surface $O'y'$ and $O'z'$, is a tangent plane to the surface at O'; and as not only the plane $AO'B$ is arbitrary, but also the position of the point O', the number of systems of oblique co-ordinate planes similar to the above is unlimited.

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158. The following problems will illustrate the principles laid down in this section.

PROB. 1. If a surface of the second order, having a center, be cut by a plane which always passes through a fixed point, to find the locus of the centers of all the plane sections.

Let $Ax^2 + By^2 + Cz^2 = D$ be the equation to the surface, $z - c = A'(x - a) + B'(y - b)$ the equation to a cutting plane, a, b, c being the co-ordinates of the point through which it always passes; therefore, eliminating y, the equation to the projection of the section on the plane of zx is

$$Ax^2 + Cz^2 + \frac{B}{B'^2}\{z - c - A'(x - a) + B'b\}^2 = D \ldots\ldots (1).$$

Let h, k, l be the co-ordinates of the center of the section, then h and l are the co-ordinates of the center of the projection; if therefore in equation (1) we substitute $x' + h$ for x, and $z' + l$ for z, the projection will be referred to its center as origin, and therefore the coefficients of x' and z' must each vanish;

$$\therefore\ Ah + \frac{B}{B'^2}\{(h - a)A'^2 - (B'b + l - c)A'\} = 0,$$

$$Cl + \frac{B}{B'^2}\{(l - c) + B'b - A'(h - a)\} = 0.$$

Multiply the latter of these equations by A' and add it to the former,

$$\therefore\ Ah + CA'l = 0, \quad \text{or } A' = -\frac{Ah}{Cl}; \quad \text{similarly, } B' = -\frac{Bk}{Cl}.$$

But since h, k, l must satisfy the equation to the cutting plane,

$$l - c = A'(h - a) + B'(k - b);$$

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$$\therefore\ l - c + \frac{Ah}{Cl}(h - a) + \frac{Bk}{Cl}(k - b) = 0,$$

$$\text{or } Ah(h - a) + Bk(k - b) + Cl(l - c) = 0,$$

the required equation, which represents a surface of the second order similar to the proposed one, the co-ordinates of its center being $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$.

It is manifest that the two surfaces intersect in a plane of which the equation is $Aax + Bby + Ccz = D$.

Prob. 2. To find the locus of the middle points of chords of a surface of the second order that has a center, all passing through a given fixed point.

Take the given point for the origin, and two conjugate diametral planes which pass through it for the planes of zx and xy, and a plane parallel to the third conjugate plane for that of yz; then the equation to the surface will be of the form

$$ax^2 + by^2 + cz^2 + 2a''x + d = 0.$$

Let $x = mz$, $y = nz$, be the equations to any chord; then combining these equations with the former, we have

$$(am^2 + bn^2 + c)z^2 + 2a''mz + d = 0,$$

in which equation the values of z are the co-ordinates of the extremities of the chord; therefore the co-ordinate of its middle point

$$z' = -\frac{a''m}{am^2 + bn^2 + c},$$

and the other two co-ordinates of the middle point are

$$x' = mz', \qquad y' = nz';$$

Hence, eliminating m and n, the required equation to the locus is

$$ax'^2 + by'^2 + cz'^2 + a''x' = 0;$$

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which represents a surface of the second order, similar to the proposed one, and similarly situated and passing through its center and through the origin.

PROB. 3. The curve of contact of a surface of the second order, and of a cone the vertex of which is given, is a plane curve.

Let $ax^2 + by^2 + cz^2 + 2a'x + 2b'y + 2c'z + d = 0$,

be the equation to the surface which may represent all surfaces of the second order. Then the equation to the tangent plane at a point xyz, is

$$(ax + a')x' + (by + b')y' + (cz + c')z' + a'x + b'y + c'z + d = 0.$$

Now because the cone and surface touch one another, at the points of contact their tangent planes coincide; and since the tangent plane of a conical surface always passes through its vertex, if h, k, l be the co-ordinates of the vertex they must satisfy the above equation;

$$\therefore\ (ax+a')h + (by+b')k + (cz+c')l + a'x + b'y + c'z + d = 0,$$

which gives the relation among the co-ordinates x, y, z of the points of contact, and shews that they all lie in the same plane. The curve of contact is therefore a plane curve resulting from the intersection of the plane whose equation we have just found with the given surface.

COR. If the surface have a center, the line joining that point and the vertex of the enveloping cone, passes through the center of the curve of contact.

Let C (fig. 52.) be the center of the surface, A the vertex of the enveloping cone, QRS the plane of the curve of contact; and let any plane drawn through A and C cut the surface in the line of the second order PQG, and the cone in the straight lines AQ, AS; then C is the center of the section PQG, and AQ, AS are tangents to it. Also, by a property of lines of the second order, QS, a line joining the points of contact, is an ordinate to the diameter PG which passes through the

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intersection of the tangents; therefore QS is parallel to the tangent at P, and is bisected in V, and $CV.CA = CP^2$. Since therefore every chord of the curve of contact drawn through the point where a line joining the vertex of the cone and the center of the surface meets the plane of contact, is bisected in it, that point is the center of the curve of contact; also the plane of contact is parallel to the plane touching the surface at the point where the same line meets the surface, and its distance from C is determined by the equation

$$CV.CA = CP^2.$$

Similarly it may be shewn, that if a cone envelope an elliptic paraboloid, the line drawn through the vertex parallel to the axis of the surface passes through the center of the curve of contact; and the portion of it between that point and the vertex is bisected by the surface.

PROB. 4. If the plane of contact of the enveloping cone of a surface of the second order always pass through a fixed point or a fixed line, the locus of the vertex of the cone will be respectively a plane or a straight line.

Let x', y', z' be the co-ordinates of the fixed point through which the plane of contact always passes; then they must satisfy its equation,

$$\therefore (ax'+a')h+(by'+b')k+(cz'+c')l+a'x'+b'y'+c'z'+d=0,$$

which gives the relation among the co-ordinates h, k, l of the vertex of the cone, and shews that the locus of the vertex is a plane; and that that plane is the plane of contact with the given surface of a cone whose vertex is in the given fixed point, for the equation may be put under the form

$$(ah+a')x'+(bk+b')y'+(cl+c')z'+a'h+b'k+c'l+d=0.$$

Again, let $x=mz+f$, $y=nz+g$, be the equations to the line through which the plane of contact always passes. Then, in order that the plane may contain the line, we must have (Art. 23.)

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$$(ah + a')\,m + (bk + b')\,n + (cl + c') = 0,$$

$$(ah + a')\,f + (bk + b')\,g + a'h + b'k + c'l + d = 0,$$

which are the equations to the straight line in which the vertex moves, its co-ordinates being h, k, l.

PROB. 5. If the plane of contact of the enveloping cone of a surface of the second order always touch a surface of the second order, the locus of the vertex will be another surface of the second order.

Let the plane of contact always touch a surface of which the equation is

(4.) $$mx^2 + ny^2 + rz^2 + 2m'x + 2n'y + 2r'z + s = 0 \ldots\ldots (1);$$

then the equation to a plane touching this surface at a point $x\,y\,z$ is

$$(mx + m')\,x' + (ny + n')\,y' + (rz + r')\,z' + m'x + n'y + r'z + s = 0,$$

which must be identical with the equation to the plane of contact, namely,

$$(ah + a')\,x' + (bk + b')\,y' + (cl + c')\,z' + a'h + b'k + c'l + d = 0;$$

(1.) $$\therefore \frac{mx + m'}{rz + r'} = \frac{ah + a'}{cl + c'}, \quad \frac{ny + n'}{rz + r'} = \frac{bk + b'}{cl + c'}, \quad (2.)$$

$$\frac{m'x + n'y + r'z + s}{rz + r'} = \frac{a'h + b'k + c'l + d}{cl + c'}; \quad (3.)$$

and it remains to eliminate x, y, z between these equations and equation (1). The result will be found to be

Agreement between (1.), (2.), (3.) and (4.) by the elimination of x, y, z.

$$\left\{\frac{m'}{m}(ah + a') + \frac{n'}{n}(bk + b') + \frac{r'}{r}(cl + c') - (a'h + b'k + c'l + d)\right\}^2$$

$$= \left(\frac{m'^2}{m} + \frac{n'^2}{n} + \frac{r'^2}{r} - s\right)\left\{\frac{1}{m}(ah + a')^2 + \frac{1}{n}(bk + b')^2 + \frac{1}{r}(cl + c')^2\right\},$$

which represents a surface of the second order.

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Cor. If the surfaces are concentric, and

$$ax^2 + by^2 + cz^2 = 1, \quad mx^2 + ny^2 + rz^2 = 1,$$

their equations, the above result is reduced to

$$\frac{a^2}{m}h^2 + \frac{b^2}{n}k^2 + \frac{c^2}{r}l^2 = 1.$$

Prob. 6. If two surfaces of the second order have a common plane section, their other curve of intersection if it exist, will also be a plane curve.

Let the equations to the two surfaces be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + D = 0,$$

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0,$$

and suppose them to have a common section in the plane of xy; then making $z = 0$, the curves represented by the equations

$$Ax^2 + By^2 + 2C'xy + 2A''x + 2B''y + D = 0,$$

$$ax^2 + by^2 + 2c'xy + 2a''x + 2b''y + d = 0,$$

are identical; if therefore m be a constant multiplier,

$$A = ma, \quad B = mb, \quad C' = mc', \quad A'' = ma'', \quad B'' = mb'', \quad D = md.$$

But in order to determine the complete intersection of the surfaces, we must combine their equations. Therefore multiplying the latter by m, and subtracting it from the former, having regard to the above relations among the coefficients, we find

$$(C - mc)z^2 + 2(A' - ma')yz + 2(B' - mb')zx + 2(C' - mc')z = 0,$$

the equation to a surface which contains all the points common to the two proposed surfaces. But it may be decomposed into

$$z = 0, \quad (C - mc)z + 2(A' - ma')y + 2(B' - mb')x + 2(C' - mc') = 0;$$

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the first represents the assumed curve of intersection in the plane of xy; the second is the equation to a plane, and cannot therefore, when combined with either surface, give any thing except a plane curve of the second order for the other curve of intersection.

PROB. 7. If two surfaces of the second order have each a principal section in the same plane, the projection of their curve of intersection on that plane is a curve of the second order.

Suppose the plane of xy to contain a principal section of each surface, then no odd power of z can enter into their equations, which will be therefore of the form

$$Ax^2 + By^2 + Cz^2 + 2C'xy + 2A''x + 2B''y + D = 0,$$

$$ax^2 + by^2 + cz^2 + 2c'xy + 2a''x + 2b''y + d = 0,$$

and if we multiply these equations respectively by c and C and subtract, we shall find the equation to the projection of the curve of intersection of the surfaces on the plane of xy, to be of the second degree in x and y.

Hence, if the axes of two surfaces of revolution intersect, since the plane containing the axes will be a principal plane of each surface, the projection of the curve of intersection on that plane, will be a curve of the second order.

PROB. 8. Any two circular sections of a surface of the second order, provided they be not parallel to one another, are situated on the same sphere.

Since all circular sections are perpendicular to the same principal plane, (Cor. Art. 103.) let fig. 53. represent that principal plane, and let EA, $E'B'$ be the projections of any two circular sections not parallel to one another, and also their diameters. Bisect EA at right angles by IC, and let C be equidistant from E and E'; therefore C is the center of a sphere intersecting the proposed surface in

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the circle EA and passing through E', and therefore (Prob. 6.) intersecting the surface in a plane curve, that is, in a second circle passing through E'; this circle must be either $E'B'$, or $E'A'$ parallel to EA; but it cannot be $E'A'$, because two parallel circles on the same sphere have their centers in a diameter perpendicular to their planes, and here IOI' being conjugate to the chords EA, $E'A'$ cannot cut them at right angles, unless the surface be one of revolution, in which case $A'E'$ and $B'E'$ become coincident; therefore the sphere will contain $E'B'$.

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SECTION VI.

ON CYLINDRICAL, CONICAL, AND CONOIDAL SURFACES, AND ON SURFACES OF REVOLUTION.

159. In the preceding sections we have considered several instances of surfaces generated by a line, straight or curved, which so moves and changes its position or form, as constantly to pass through one or more fixed curves, or directrices. We shall now extend the same considerations to the general case of a generating curve represented by the equations

$$f(x, y, z, \alpha, \beta) = 0, \quad f_1(x, y, z, \alpha, \beta) = 0 \ldots\ldots (1),$$

containing two variable parameters, α, β, and subject to pass through one fixed curve, or directrix. The *species* of the generating curve is determined, because the functions f, f_1, are supposed to be known, but its position and dimensions will change corresponding to the different values of α and β; and it will generate a *solid* if the parameters α and β vary independently of one another. Thus the circle represented by

$$z^2 + (y - \beta)^2 = c^2 - \alpha^2, \quad x - \alpha = 0,$$

gives, by the elimination of α, the equation

$$z^2 + x^2 + (y - \beta)^2 = c^2,$$

which represents a sphere, radius c, and center in the axis of y; and if β receive all values from 0 to $\pm\infty$, the equation will belong to all the points of a *solid* cylinder, radius c, and axis coinciding with the axis of y. But if we suppose the generating line (1) in every position to have a

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point in common with a fixed curve or directrix represented by the equations

$$y = \phi(x), \quad z = \psi(x),$$

then the equations to the generating line and directrix must be simultaneously satisfied by the same system of values for x, y, z belonging to that common point; if therefore we eliminate x, y, z between them, we shall have $\beta = F(\alpha)$, the relation between the parameters, in order that the generating line may meet the directrix. Hence, the generating line in any position will be represented by the system of equations

$$f(x, y, z, \alpha, \beta) = 0, \quad f_1(x, y, z, \alpha, \beta) = 0, \quad \beta = F(\alpha).$$

If therefore we eliminate α and β between them, that is, if we determinine α and β from the two former in terms of x, y, z, so that $\beta = u$, $\alpha = v$, and substitute in the latter, we shall have $u = F(v)$ for the equation to the surface; and we observe that u and v, do not change for surfaces of the same family, that is, for those which admit the same generating line, but that F, which depends upon ϕ and ψ, changes for each individual surface of the family.

These considerations we shall now apply to several of the more common cases, where the generating line is a straight line, or a circle.

160. To find the general equation to cylindrical surfaces.

A cylindrical surface is generated by a straight line which moves parallel to itself, and always passes through a given curve.

Let the equations to the generating line in any position be $x = mz + \alpha$, $y = nz + \beta$, α and β being variable quantities depending upon that position, and m and n constant quantities, since the line is always parallel to itself. Also let $y = \phi(x)$, $z = \psi(x)$, be the equations to the di-

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rectrix, or curve through which the generating line always passes, and x', y', z' the co-ordinates of the point in which they meet; then x', y', z' must satisfy both the equations to the directrix, and generating line;

$$\therefore\ x' = mz' + a, \quad y' = nz' + \beta, \quad y' = \phi(x'), \quad z' = \psi(x');$$

now by means of these four equations, we may eliminate x', y', z', and there will remain $\beta = F(\alpha)$; but $\beta = y - nz$, $\alpha = x - mz$;

$$\therefore\ y - nz = F(x - mz),$$

a relation among the co-ordinates of any point in the generating line, and therefore the equation to the surface which it describes. In this case the quantities u and v are $y - nz$, and $x - mz$, which remain the same for all cylinders, whilst the function F will alter with the different directrices employed.

Cor. If for the purpose of eliminating the arbitrary function, we differentiate the equation $y - nz = F(x - mz)$, successively with respect to x and y, we find

$$-np = F'(x - mz)(1 - mp), \quad 1 - nq = F'(x - mz)(-mq);$$

therefore, dividing one result by the other,

$$\frac{np}{1 - nq} = \frac{1 - mp}{mq}, \quad \text{or } md_x z + nd_y z = 1,$$

which is the differential equation to cylindrical surfaces; we may however obtain it more easily by the consideration of the tangent plane, as follows.

161. To find the differential equation to cylindrical surfaces.

One of the distinguishing properties of cylindrical surfaces is, that the tangent plane, since it always contains a generating line, (Cor. Art. 124.) is always parallel to a fixed straight line.

Let $x' = mz'$, $y' = nz'$ be the equations to a line through the origin, to which the generating line is always parallel; then this line is parallel to the tangent plane, whose equation is

$$z' - z = p(x' - x) + q(y' - y);$$

$\therefore$ (Art. 23.) $mp + nq = 1$, or $md_x z + nd_y z = 1$, as before.

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Cor. This equation may be employed to discover whether the surface represented by a proposed equation $u = 0$, is cylindrical or not. For, obtaining the values of p and q, as in Cor. Art. 107, and substituting, we have

$$md_{(x)}u + nd_{(y)}u + d_{(z)}u = 0,$$

which must be satisfied for all points of the surface, that is, for all values of x, y, z, if the surface be cylindrical; hence we must equate to zero the coefficients of the different powers of the co-ordinates in it, and examine whether the resulting conditions can be satisfied by real values of m and n.

162. Given the direction of the generating line, to find the equation to the cylindrical surface which envelopes a given curve surface.

We have seen that whatever be the nature of the directrix, the equation $md_x z + nd_y z = 1$, must subsist between the differential coefficients $d_x z$, $d_y z$, derived from the equation to a cylindrical surface. But at the points where the cylinder touches the surface, the values of $d_x z$, $d_y z$, are the same for both; therefore at those points the above relation subsists between the differential coefficients derived from the equation to the surface; that is, the co-ordinates of the points of contact are such that the equation $md_x z + nd_y z = 1$, is satisfied. Hence if we differentiate $u = 0$ the given equation to the surface, and write the values thence obtained of $d_x z$, $d_y z$ in the above equation, the result together with the equation to the surface, will be the equations to the directrix; and we know the direction of the generating line, and therefore can find the equation to the required surface by Art. 160.

163. To find the general equation to conical surfaces.

A conical surface is generated by a straight line which passes through a given point, and always meets a given curve.

Let the co-ordinates of the given point or vertex be a, b, c;

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$$\therefore\ x - a = \alpha(z - c), \quad y - b = \beta(z - c),$$

are the equations to the generating line in any position, α and β being variable quantities depending upon that position. Also let $y = \phi(x)$, $z = \psi(x)$, be the equations to the directrix or curve through which the generating line always passes, and x', y', z', the co-ordinates of the point in which they meet; then x', y', z' must satisfy both the equations to the generating line and directrix;

$$\therefore\ x' - a = \alpha(z' - c),\ y' - b = \beta(z' - c),\ y' = \phi(x'),\ z' = \psi(x').$$

Now, by means of these four equations, we may eliminate x', y', z', and there will remain $\beta = F(\alpha)$;

$$\text{but}\ \beta = \frac{y - b}{z - c},\quad \alpha = \frac{x - a}{z - c};$$

$$\therefore\ \frac{y - b}{z - c} = F\left(\frac{x - a}{z - c}\right),$$

a relation among the co-ordinates of any point in the generating line, and therefore the equation to the surface which it describes. When the vertex is situated in the origin, the equation is reduced to $\frac{y}{z} = F\left(\frac{x}{z}\right)$, which expresses that the equation is *homogeneous* in x, y, z.

Cor. If from the above equation we eliminate the arbitrary function by differentiation, as in Cor. Art. 160, we find

$$z - c = d_x z(x - a) + d_y z(y - b),$$

which is the differential equation to conical surfaces; but which may be obtained more easily by the consideration of the tangent plane, as in the following Article.

164. To find the differential equation to conical surfaces. The distinguishing property of conical surfaces is, that the tangent plane, since it always contains one of the generating lines (Cor. Art. 124.), always passes through the vertex.

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Let a, b, c denote the co-ordinates of the vertex, then they must satisfy the equation to the tangent plane at any point xyz; that is, $z' - z = p(x' - x) + q(y' - y)$ must be satisfied by $x' = a$, $y' = b$, $z' = c$; therefore, whatever be the nature of the directrix, the differential coefficients $d_x z$, $d_y z$, derived from the equation to the surface, must be such as to satisfy the equation

$$z - c = p(x - a) + q(y - b),$$

which is the differential equation to conical surfaces.

Cor. As in cylindrical surfaces (Art. 161.), this equation, when put under the form

$$(x - a)\,d_{(x)}u + (y - b)\,d_{(y)}u + (z - c)\,d_{(z)}u = 0,$$

may be employed to discover whether a proposed equation $u = 0$ represents a conical surface or not; a, b, c being the unknown quantities to which must be applied what was there said relative to m and n.

165. Given the position of the vertex, to find the equation to the conical surface which envelopes a given curve surface.

Whatever be the nature of the directrix, we have seen that the differential coefficients derived from the equation to a conical surface must satisfy the equation

$$z - c = d_x z\,(x - a) + d_y z\,(y - b);$$

but at the points where the cone touches the surface, the values of $d_x z$, $d_y z$, are the same for both; therefore at those points the above relation subsists between the differential coefficients derived from the equation to the surface, that is, the co-ordinates of the points of contact are such, that the above equation is satisfied. Hence if we obtain values of $d_x z$, $d_y z$, from the given equation to the surface, and substitute them in the above equation, the result, together with the equation to the surface, will be the equations to the curve of contact of the two surfaces, or

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to the directrix; and we know the co-ordinates of the vertex, and can therefore find the equation to the required conical surface by Art. 163. If the vertex of the cone be considered as a luminous point, the curve of contact whose equations we have just found, is that which on the surface separates the illumined and obscure parts; if it be considered as the place of the eye, the curve of contact is the line of the apparent contour of the surface.

166. To find the general equation to conoidal surfaces.

A conoidal surface is generated by a straight line which moves parallel to a given plane, and always meets a given fixed straight line, and a given curve.

Let the given plane, called the directing plane, be taken for that of xy, and the point in which the straight directrix OA (fig. 54.) meets it, for the origin; then the equations to OA will be $x = mz$, $y = nz$. Also let the equations to the curvilinear directrix BC be $y = \phi(x)$, $z = \psi(x)$; and let $z = \beta$, $y = \alpha x + \gamma$ be the equations to the generating line DC in any position, since it is always parallel to the plane of xy; then since the generating line always meets OA, we must have

$$n\beta = \alpha m\beta + \gamma;$$

therefore, eliminating γ, the equations to the generating line become

$$z = \beta, \quad y - n\beta = \alpha(x - m\beta);$$

we must next express that it meets the curve BC, whose equations are $y = \phi(x)$, $z = \psi(x)$; therefore, eliminating x, y, z, we have $\beta = F(\alpha)$; hence the generating line in any position will be represented by the system of equations

$$z = \beta, \quad y - n\beta = \alpha(x - m\beta), \quad \beta = F(\alpha);$$

therefore, eliminating α and β, the equation to the surface generated is

$$z = F\left(\frac{y - nz}{x - mz}\right).$$

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Cor. 1. This surface is *twisted*, that is, no two consecutive generating lines are in the same plane; for the line DC in passing to the consecutive position $D'C'$ may be supposed to glide along the tangent CT; therefore, in order that DC and $D'C'$ may be in the same plane, OA and CT must be in the same plane, which cannot happen, at least for all the tangents, unless the curve BC is entirely in the same plane with AO, in which case the surface generated will be a plane.

If the straight directrix be perpendicular to the directing plane, $m = 0$, $n = 0$, and the equation to the surface, which is then called a *right* conoid, becomes $z = F\left(\frac{y}{x}\right)$; at which we may readily arrive by a direct investigation, as the equations to the generating line will be $z = \beta$, $y = \alpha x$. Also it is manifest that the equation to the oblique conoid will assume the form $z = F\left(\frac{y}{x}\right)$, if we refer it to a system of oblique co-ordinate axes, of which OA is one, and any two lines drawn through O in the directing plane are the others.

Cor. 2. If from the equation $z = F\left(\frac{y - nz}{x - mz}\right)$ we eliminate the arbitrary function by differentiation, as in the former cases, we find

$$p(x - mz) + q(y - nz) = 0,$$

the differential equation to conoidal surfaces; at which we may arrive more easily as follows.

167. To find the differential equation to conoidal surfaces.

Their characteristic property is that the tangent plane at any point contains the generating line which passes through that point, (for it contains the tangent lines to all curves traced on the surface through that point, and a straight line is its own tangent,) that is, a line parallel to the plane

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of xy, and intersecting another line whose equations are $x' = mz'$, $y' = nz'$. Now the equation to the tangent plane at a point xyz is

$$z' - z = p(x' - x) + q(y' - y),$$

and if we intersect it by a plane parallel to xy at a distance z from it, their line of intersection will have for its equations

$$z' = z, \quad p(x' - x) + q(y' - y) = 0,$$

and since this is coincident with the generating line, it will intersect the rectilinear directrix of which the equations are $x' = mz'$, $y' = nz'$;

$$\therefore\ p(x - mz) + q(y - nz) = 0,$$

which is the relation that must be satisfied by the co-ordinates of every point in the surface.

For a right conoid, or for an oblique conoid referred to the rectilinear directrix as one of the axes, the differential equation is

$$px + qy = 0.$$

168. Having given the rectilinear directrix, to find the equation to the conoidal surface which envelopes a given curve surface.

As in the former cases, if $u = 0$ be the equation to the given surface, the curve of contact which forms the curvilinear directrix, will be given by the equations

$$u = 0, \quad (x - mz)\,d_{(x)}u + (y - nz)\,d_{(y)}u = 0,$$

which must be employed instead of the equations $y = \phi(x)$, $z = \psi(x)$; the remainder of the process will be the same as that explained in Art. 166.

169. To find the general equation to surfaces of revolution.

A surface of revolution is generated by a curve revolving about a fixed axis; if the curve be situated entirely in a

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plane passing through the axis, then every section of the surface through the axis, which is called a *meridian*, will reproduce the generating curve; otherwise, the section will be different from the generating curve. Also every section perpendicular to the axis, which is called a *parallel* of the surface, will be a circle whose center is in the axis and plane perpendicular to it, and whose perimeter passes through the generating curve; hence a surface of revolution may be supposed to be generated by a circle whose center moves along a fixed straight line to which its plane is perpendicular, and whose perimeter always passes through a given curve. The advantages of the second mode of generation are, that the moveable circle is a generating curve of an invariable nature and common to all surfaces of the class, whilst the curve through which it always passes is a variable directrix which changes for each particular surface, agreeably to the mode of generation occurring in every instance in this section.

Let $x - a = m(z - c)$, $y - b = n(z - c)$, be the equations to the axis or fixed straight line along which the center of the generating circle moves, and which we suppose to be drawn through a given point (a, b, c) in a known direction. Then every one of the parallels of the surface may be regarded as the intersection of a plane perpendicular to this axis, with a sphere whose center is any fixed point in the axis, the point (a, b, c), for instance, and whose radius varies so as to give the section of the proper magnitude; consequently the equations to the generating circle will be

$$\left.\begin{aligned} z + mx + ny &= \beta, \\ (x-a)^2 + (y-b)^2 + (z-c)^2 &= \alpha. \end{aligned}\right\} \ldots\ldots (1).$$

Let $y = \phi(x)$, $z = \psi(x)$, be the equations to the directrix or curve which the perimeter of the generating circle always meets; then these four equations will be satisfied by the same system of values of x, y, z, which are the co-ordinates of the point of intersection; hence, eliminating x, y, z from them, we find $\beta = F(\alpha)$ the condition to be joined to the equations (1) in order that the circle which

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they represent may really be a parallel of the surface; hence, eliminating a and β from the equations to the parallel, we find

$$z + mx + ny = F\{(x-a)^2 + (y-b)^2 + (z-c)^2\},$$

a relation among the co-ordinates of the generating circle in any position, and therefore the equation to the surface which it describes.

Cor. 1. If the axis of revolution coincide with the axis of z, and we take the center of the sphere for origin, we have

$$m = n = 0, \quad a = b = c = 0, \quad \text{and} \quad z = F(x^2 + y^2 + z^2),$$

$$\text{or} \quad z = f(x^2 + y^2),$$

as we have already seen, Art. 52. In this case it is more convenient to regard each parallel as the intersection of a right cylinder with a plane perpendicular to it, that is, instead of equations (1), to use

$$z = \beta, \quad x^2 + y^2 = a,$$

which joined to the usual relation $\beta = F(a)$, give immemediately

$$z = F(x^2 + y^2).$$

Cor. 2. If we differentiate the above equation with respect to x and y, and divide one result by the other, so as to eliminate the arbitrary function, we find

$$\frac{m+p}{n+q} = \frac{x-a+(z-c)p}{y-b+(z-c)q},$$

$$\text{or } p\{y-b-n(z-c)\} - q\{x-a-m(z-c)\} = n(x-a) - m(y-b),$$

the differential equation to surfaces of revolution, at which we may also arrive as follows.

170. To find the differential equation to surfaces of revolution.

Their distinguishing property is, that the normal always meets the axis of revolution. This will be readily seen if

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we consider that the tangent plane at any point necessarily contains the line touching the parallel at that point, and is therefore perpendicular to the meridian plane passing through that point; consequently the normal will lie in the meridian plane, and therefore meet the axis.

The equations to the normal at a point xyz are

$$x' - x + p(z' - z) = 0, \quad y' - y + q(z' - z) = 0;$$

let, as above, the equations to the axis be

$$x' - a - m(z' - c) = 0, \quad y' - b - n(z' - c) = 0,$$

then if these two lines intersect, their equations must be satisfied by the same values of x', y', z', which will be the co-ordinates of their point of intersection. Hence if we eliminate x', y', z', the result will be the differential equation to the surface. Subtracting, we find

$$m(z' - c) + p(z' - z) - x + a = 0, \quad n(z' - c) + q(z' - z) - y + b = 0,$$

$$\text{or, } (m + p)(z' - z) + m(z - c) - x + a = 0,$$

$$(n + q)(z' - z) + n(z - c) - y + b = 0;$$

$\therefore$ eliminating $z' - z$, we find as before

$$(m + p)\{n(z - c) - y + b\} = (n + q)\{m(z - c) - x + a\} \ldots (1),$$

which expresses that a surface is one of revolution about an axis passing through a point a, b, c, and having a direction determined by the quantities m and n.

If the axis of revolution coincide with the axis of z, the equation becomes

$$py - qx = 0.$$

Cor. The above equation will enable us to ascertain whether a proposed equation $u = 0$, represents a surface of revolution; for, deducing the values of p and q, and substituting them in (1), the result ought to be true for all values of x, y, z. Hence we must equate to zero the coefficients of the different powers of the co-ordinates, and we shall obtain among the unknown quantities a, b, c, m, n

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a certain number of equations, which must be consistent with one another, in order that the surface may be one of revolution. The process applied to the general equation of the second order will furnish the conditions already obtained, (Art. 149.), by a different method.

171. Having given the position of the axis, to find the equation to the surface of revolution which shall envelope a given curve surface.

As in the similar cases which have preceded, we must begin by finding the equations to the curve of contact which will be the directrix of the moveable circle. Let $u = 0$ be the equation to the proposed surface, then at all points along the curve of contact the tangent planes of the two surfaces are coincident; and therefore the values of p and q derived from $u = 0$, and substituted in (1), must satisfy it for all points in the curve of contact; the result of this substitution together with $u = 0$, will therefore be the equations to the directrix, and then the equation to the required surface of revolution may be found as before.

It is manifest that we thus determine a surface which, if the proposed surface revolved about the given axis to which it was fixed, would touch and envelope it in every position.

172. The above are the principal propositions relative to the commoner families of surfaces which admit only one directrix, or only one of a variable form; for though Conoidal surfaces have two directrices, yet one of them, namely, the rectilinear one, is constant for all surfaces of the family. Consequently, for Conoidal, as well as for all the other families of surfaces considered in this section, the finite equation involves only one arbitrary function, and the differential equation cleared of the arbitrary function, is only of the first order. We shall now illustrate these results by several applications.

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I. To find the equation to the cylindrical surface which envelopes a given oblate spheroid.

Let the center of the spheroid be the origin, and the plane of zx that which contains the axis of the spheroid, and a line through the origin to which the generating line of the cylinder is always parallel; therefore the equations to the generating line in any position will be

$$x = mz + a, \quad y = \beta.$$

Let $x^2 + y^2 + \frac{a^2}{c^2} z^2 = a^2$ be the equation to the spheroid,

$\therefore$ the equation $m d_z z + n d_y z = 1$ becomes, since $n = 0$,

$$m \frac{c^2}{a^2} \frac{x}{z} + 1 = 0.$$

Hence the equations to the directrix are

$$c^2 (x^2 + y^2) + a^2 z^2 = a^2 c^2, \quad m c^2 x + a^2 z = 0,$$

between which, and the equations $x = mz + a$, $y = \beta$, eliminating x, y, z we find

$$\beta^2 (a^2 + m^2 c^2) + a^2 a^2 = a^2 (a^2 + m^2 c^2).$$

Therefore, restoring the values of α and β, the equation to the surface is

$$y^2 (a^2 + m^2 c^2) + a^2 (x - mz)^2 = a^2 (a^2 + m^2 c^2).$$

If we make $z = -c$, we shall obtain the equation to the curve in which the cylinder meets the horizontal plane on which the spheroid stands; this gives

$$y^2 (a^2 + m^2 c^2) + a^2 (x + mc)^2 = a^2 (a^2 + m^2 c^2),$$

$$\text{or,} \quad \frac{y^2}{a^2} + \frac{(x + mc)^2}{a^2 + m^2 c^2} = 1,$$

which is the equation to an ellipse, the origin, that is, the point on which the spheroid stands, being the focus. Hence, an oblate spheroid placed in the Sun on a horizontal plane stands in the focus of its shadow.

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II. If a cylindrical surface be circumscribed about a surface of the second order of which the equation is $Ax^2 + By^2 + Cz^2 = 1$, and if R be the length of the semi-diameter to which the cylinder is parallel, and λ, μ, ν the angles which it makes with the axes, the equation to the surface is

$$Ax^2 + By^2 + Cz^2 - 1 = R^2(Ax\cos\lambda + By\cos\mu + Cz\cos\nu)^2.$$

III. To find the equation to the conical surface which envelopes a given surface of the second order.

Let the equation to a given surface be

$$Ax^2 + By^2 + Cz^2 = D,$$

then the equation $z - c = p(x - a) + q(y - b)$ gives

$$Ax(x - a) + By(y - b) + Cz(z - c) = 0 \ldots\ldots(1),$$

which, by virtue of the equation to the surface, becomes

equation of a plane curve. $$Aax + Bby + Ccz = D \ldots\ldots(2).$$

Therefore, the curve of contact will be determined by equations (1) and (2); and eliminating x, y, z between them and the equations

$$x - a = \alpha(z - c), \quad y - b = \beta(z - c),$$

we have, first, from (1), $Aax + B\beta y + Cz = 0$,

and combining this with (2) and the equations to the generating line, we easily effect the elimination and obtain for result

$$(Aa^2 + B\beta^2 + C)(Aa^2 + Bb^2 + Cc^2 - D) = (Aa\alpha + Bb\beta + Cc)^2.$$

Hence, restoring the values of α and β, we have for the equation to the surface

$$\{A(x - a)^2 + B(y - b)^2 + C(z - c)^2\}\{Aa^2 + Bb^2 + Cc^2 - D\}$$
$$= \{Aa(x - a) + Bb(y - b) + Cc(z - c)\}^2,$$

or, reducing,

$$(Ax^2 + By^2 + Cz^2 - D)(Aa^2 + Bb^2 + Cc^2 - D)$$
$$= (Aax + Bby + Ccz - D)^2.$$

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Hence, the cone touches the surface along the curve in which it is intersected by the plane represented by (2). Also if $S = AC$ (fig. 52.), the length of the line joining the vertex of the cone and the origin, and $R = CP$ the portion of that line which forms a semidiameter of the surface, it may be easily shewn that

$$Aa^2 + Bb^2 + Cc^2 = D \cdot \frac{S^2}{R^2};$$

for $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ in this case,

which may be substituted in the above equation to the conical surface.

Cor. If instead of referring the surface to its axes, we refer it to a system of conjugate diameters, one of which passes through the given vertex and is taken for the axis of z, the equations to the generating line will be $x = \alpha(z - c)$ $y = \beta(z - c)$, and the equations to the curve of contact

$$Ax^2 + By^2 + Cz^2 = D, \quad Ccz = D;$$

$\therefore$ eliminating x, y, z, we find $(A\alpha^2 + B\beta^2)\left(c^2 - \frac{D}{C}\right) = D$,

and restoring the values of α and β, the equation to the surface is

$$(z - c)^2 = \left(\frac{c^2}{D} - \frac{1}{C}\right)(Ax^2 + By^2).$$

IV. One of the most useful applications of the general method of determining conical surfaces, is to find the stereographic projection of a curve upon any plane; that is, the trace upon the plane, of a conical surface, called the optical cone, whose vertex is the place of the eye, and directrix the given curve.

If we take a line drawn through the place of the eye, perpendicular to the plane of projection for the axis of z, the equation to the optical cone will be

$$\frac{y}{z + c} = F\left(\frac{x}{z + c}\right),$$

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which results from the general equation to conical surfaces, by making $a = b = 0$, and changing the sign of c, that is, supposing the vertex to be situated below the plane of xy. If in this equation we make z constant, we obtain the equation to the stereographic projection of the given curve upon any plane perpendicular to the axis of z; and if in the result we make c infinite (so that the projecting lines may all become parallel to one another and perpendicular to the plane of projection), we shall find the equation to the orthographic projection, which is the sort of projection so frequently employed in the preceding part of this work. When the curves to be projected are situated on the surface of a sphere, the term *stereographic* is usually confined to the case where the eye is situated in the pole of the great circle whose plane is the plane of projection; and when the eye is in the center, and the plane of projection is a tangent plane to the surface, the projection is called *gnomonic*. We shall now illustrate the method by the following examples.

1. To find the stereographic, gnomonic, and orthographic projections of any circle of a sphere.

Let the center of the sphere be the origin, and a plane perpendicular to the plane of the circle to be projected, the plane of zx; then

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= r^2 \\ z &= ax + b \end{aligned}\right\}$$

are the equations to a circle of the sphere, forming the directrix of the optical cone;

$$\text{also } x = \alpha(z + c), \quad y = \beta(z + c),$$

are the equations to the generating line; hence, eliminating x, y, z, between these four equations, we find

$$z = a\alpha(z + c) + b, \quad \text{or } z = \frac{ac\alpha + b}{1 - a\alpha},$$

$$x = \alpha(z + c) = \alpha\frac{b + c}{1 - a\alpha}, \quad y = \beta\frac{b + c}{1 - a\alpha},$$

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$$\therefore (b+c)^2(\alpha^2+\beta^2) + (ac\alpha + b)^2 = r^2(1-a\alpha)^2;$$

or, restoring the values of α and β,

$$(b+c)^2(x^2+y^2) + \{acx + b(z+c)\}^2 = r^2(z+c-ax)^2,$$

the equation to the optical cone, from which those to the projections may be deduced.

1. In stereographic projection, the plane of projection passes through the center, and the eye is situated in the surface of the sphere, therefore $z = 0$, $c = r$;

$$\therefore (b+r)^2(x^2+y^2) + r^2(ax+b)^2 = r^2(r-ax)^2,$$

$$\text{or } x^2 + y^2 + \frac{2ar^2}{r+b}x = r^2 \cdot \frac{r-b}{r+b};$$

the equation to a circle, whose radius

$$= \frac{r}{r+b}\sqrt{r^2(1+a^2) - b^2},$$

and distance of its center from that of the sphere $= \frac{ar^2}{r+b}$.

2. In gnomonic projection, the plane of projection touches the surface, and the eye is situated in the center of the sphere, therefore $z = r$, $c = 0$;

$$\therefore b^2(x^2+y^2) + b^2r^2 = r^2(r-ax)^2,$$

$$\text{or } x^2(b^2 - a^2r^2) + b^2y^2 = r^2(r^2-b^2) - 2r^3ax;$$

the equation to an ellipse, hyperbola, or parabola, according as $b^2 >, <,$ or $= a^2r^2$.

3. In orthographic projection $c = \infty$, $z = 0$;

$$\therefore x^2 + y^2 + (ax+b)^2 = r^2,$$

the equation to an ellipse.

In the same manner it may be shewn that the stereographic projection of any plane section of a paraboloid of revolution, upon a plane perpendicular to its axis, is a circle,

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the eye being placed in the vertex; or that the stereographic projection of any section of an oblate spheroid, upon the plane of the equator, is a circle, the eye being placed in the pole.

We may employ the same method also to compare any angle on the surface of a sphere with its different projections; this, however, may be done more simply by geometrical considerations, as will be seen in the following problem.

PROB. The gnomonic, orthographic, and stereographic projections of the rhumb-line on the surface of a sphere, are three of Cotes's spirals.

The rhumb-line is a curve of double curvature, traced on the surface of a sphere so as to cut all meridians under the same angle.

1. To find the gnomonic projection.

Let yBx, (fig. 55.) be the plane of projection perpendicular to BC, Tt its intersection with a plane touching the sphere at a point P in the plane of zx, then Tt is parallel to By; Pt a tangent to the rhumb-line at P, Q the projection of P determined by producing CP to meet the plane xBy. Then $\angle TQt$ is the projection of TPt;

$$\text{and } \tan TQt = \frac{Tt}{TQ} = \frac{TP}{TQ} \cdot \frac{Tt}{TP} = \frac{TP}{TQ} \tan TPt = \frac{CB}{CQ} \tan a,$$

if a = the constant angle at which the rhumb-line cuts the meridian. Let $BQ = \rho$, p = perpendicular from B on Qt which touches the projection of the rhumb-line at Q, $BC = a$,

$$\text{then } \frac{p}{\sqrt{\rho^2 - p^2}} = \frac{a \tan a}{\sqrt{\rho^2 + a^2}}, \quad \text{or } \frac{\rho^2}{p^2} - 1 = (\cot a)^2 \left(\frac{\rho^2}{a^2} + 1\right),$$

$$\text{or } \frac{1}{p^2} = \frac{(\operatorname{cosec} a)^2}{\rho^2} + \frac{(\cot a)^2}{a^2},$$

the equation to one of Cotes's spirals.

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2. To find the orthographic projection.

Let xCy be the plane of projection (fig. 56.), Tt its intersection with a plane touching the sphere at a point P, situated in the plane of xx, then Tt is parallel to Cy; Pt a tangent to the rhumb-line at P, N the projection of P, then $\angle TNt$ is the projection of TPt;

$$\text{and } \tan TNt = \frac{Tt}{TN} = \frac{TP}{TN} \cdot \frac{Tt}{TP} = \frac{TP}{TN} \cdot \tan TPt = \frac{CP}{PN} \tan \alpha;$$

or if $CN = \rho$, p = perpendicular from C on Nt which touches the projection at N,

$$\frac{p}{\sqrt{\rho^2 - p^2}} = \frac{a \tan \alpha}{\sqrt{a^2 - \rho^2}}; \quad \therefore \frac{\rho^2}{p^2} - 1 = (\cot \alpha)^2 \left(1 - \frac{\rho^2}{a^2}\right),$$

$$\text{or } \frac{1}{p^2} = \frac{(\operatorname{cosec} \alpha)^2}{\rho^2} - \frac{(\cot \alpha)^2}{a^2},$$

the equation to the hyperbolic spiral, as already found p. 131.

3. To find the stereographic projection.

Let $CO = CP$ be perpendicular to the plane of projection xCy (fig. 56.); join PO meeting the plane of projection in Q, which is therefore the projection of P, and $\angle TQt$ of TPt. Then $\angle TPQ$ = complement of CPO = complement of COP = PQT, $\therefore PT = QT$; and since Tt is perpendicular both to PT and QT, $\therefore \angle TQt = TPt = \alpha$; therefore since the angle between the tangent and radius vector is constant, the curve is an equiangular spiral, its equation being $p = \rho \sin \alpha$.

Cor. It appears, that if an angle α on the surface of a sphere have one of the arcs which contain it, a meridian, and if γ denote its gnomonic, ω its orthographic projection, and θ the arc of the containing meridian intercepted between its vertex and a plane through the center parallel to the plane of projection, then

$$\tan \gamma = \sin \theta \tan \alpha, \quad \tan \omega = \frac{1}{\sin \theta} \tan \alpha;$$

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also $\tan \alpha = \sqrt{\tan \omega . \tan \gamma}$, that is, the tangent of the stereographic projection is a mean proportional between the tangents of the orthographic and gnomonic projections of the same angle.

V. To find the equation to the conoidal surface generated by a horizontal straight line, which constantly passes through a Helix, and the axis of the vertical cylinder on which it is traced.

The equations to the helix are (Prob. 2, Art. 132.)

$$x^2 + y^2 = a^2, \quad z = na \cos^{-1} \frac{x}{a};$$

let the equations to the generating line be $z = \beta$, $y = ax$, then eliminating x, y, z, $x^2 + a^2x^2 = a^2$,

$$\text{or } \frac{x}{a} = \frac{1}{\sqrt{1+a^2}}; \quad \therefore \beta = na \cos^{-1} \frac{1}{\sqrt{1+a^2}} = na \tan^{-1} a;$$

therefore, restoring the values of a and β, the equation to the surface is

$$z = na \tan^{-1} \frac{y}{x}.$$

This is the surface presented by the inferior superficies of a staircase, attached to a vertical column round which it winds.

VI. To find the equation to the surface generated by the revolution of any straight line about a fixed axis, but not in the same plane.

Let the fixed axis be taken for the axis of z, and let the equations to the straight line be

$$x = Az + h, \quad y = Bz + k,$$

and the equations to the generating circle

$$z = \beta, \quad x^2 + y^2 = a;$$

therefore eliminating x, y, z, we have

$$(A\beta + h)^2 + (B\beta + k)^2 = a,$$

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or, restoring the values of α and β,

$$(Az + h)^2 + (Bz + k)^2 = x^2 + y^2,$$

$$\text{or} \quad x^2 + y^2 - (A^2 + B^2)z^2 - 2(Ah + Bk)z = h^2 + k^2,$$

which is the equation to the surface, and manifestly represents a hyperboloid of one sheet with its center in the axis of z. If in this equation we make $x = 0$, or $y = 0$, in order to find the equation to a section through the axis, the resulting equation represents a hyperbola; therefore the surface is a hyperboloid of revolution. Also if, retaining one of the above equations to the directrix, we change the signs of the second member in the other, the equation to the surface is not altered; therefore the same surface may be generated by two different straight lines, and through any one of its points two straight lines may be drawn so as to entirely coincide with it. This agrees with Art. 74.

If we take for axis of x the shortest distance of the directrix from the fixed axis, the equations to the directrix, since it is parallel to the plane of yz, will be $x = h$, $y = Bz$; therefore putting $A = 0$, $k = 0$, the equation to the surface is

$$x^2 + y^2 - B^2z^2 = h^2,$$

the origin being the center.

VII. An oblate spheroid revolves about any diameter; to find the equation to the surface which envelopes it in every position. This will be a surface of revolution.

Let $x = mz$, $y = nz$ be the equations to the diameter,

$c^2(x^2 + y^2) + a^2z^2 = a^2c^2$ the equation to the spheroid;

then the equation $(y - nz)p - (x - mz)q + my - nx = 0$ becomes

$$(my - nx)(a^2 - c^2) = 0;$$

therefore the equations to the directrix are

$$c^2(x^2 + y^2) + a^2z^2 = a^2c^2, \quad nx = my,$$

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and the equations to the generating circle,

$$z + mx + ny = \beta, \quad x^2 + y^2 + z^2 = a;$$

eliminating x, y, z between these four equations, we find

$$a^2(a - c^2)(m^2 + n^2) = (\beta\sqrt{a^2 - c^2} - c\sqrt{a^2 - a})^2;$$

therefore, restoring the values of a and β, the required equation is

$$a^2(x^2 + y^2 + z^2 - c^2)(m^2 + n^2)$$

$$= \{(z + mx + ny)\sqrt{a^2 - c^2} - c\sqrt{a^2 - x^2 - y^2 - z^2}\}^2.$$

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SECTION VII.

ON SURFACES HAVING MORE THAN ONE ARBITRARY CURVILINEAR DIRECTRIX, AND ON DEVELOPABLE SURFACES, AND ENVELOPES.

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173. In the preceding Section we considered several surfaces where the equations to the generating curve contained two variable parameters; and the necessary dependence of these parameters upon one another, in order that a surface and not a solid might be generated, was established by making the generating curve in every position have a point in common with a given fixed directrix. The resulting equation to the surface consequently involved but one arbitrary function; and the differential equation, characterizing the nature of the surface independent of that directrix, was of the first order. We shall now take a more general view of the subject of the generation of surfaces, and suppose the equations to the generating curve

$$f(x, y, z, a, \beta, \gamma, \&c.) = 0, \quad f_1(x, y, z, a, \beta, \gamma, \&c.) = 0,$$

to contain any number n of variable parameters a, β, γ, &c. This curve will not, by passing through all the various positions and forms corresponding to the various values of the parameters, generate a determinate surface, unless we subject it to such conditions as will leave only one of the parameters, a for instance, arbitrary; and this may be done by making it, in every position, pass through $n - 1$ curves represented by the equations

$$y = \phi(x), \quad z = \psi(x), \quad y = \phi_1(x), \quad z = \psi_1(x), \ \&c.$$

For in order that it may have a point in common with the first directrix, its two equations and the equations $y = \phi(x)$,

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$z = \psi(x)$, must be satisfied by the same system of values of x, y, z, which are the co-ordinates of that point; therefore, eliminating x, y, z between these four equations, we find $F(\alpha, \beta, \gamma, \&c.) = 0$, which expresses the necessary dependence among the parameters in order that the generating curve may pass through the first directrix. In the same manner, each directrix will furnish a fresh relation between α, β, γ, &c.; so that to represent completely the generating curve passing through the $n-1$ directrices, we must join to its two equations, the $n-1$ equations of condition $F(\alpha, \beta, \gamma, \&c.) = 0$, $F_1(\alpha, \beta, \gamma, \&c.) = 0$, &c.; and if from these latter we suppose β, γ, &c. to be all determined in terms of α, and substitute them in the equations to the generating curve, those equations will then involve only one parameter, by the elimination of which between them we shall arrive at an equation containing no arbitrary quantity, and representing a determinate surface generated by the moveable curve. Hence the final equation will contain as many arbitrary functions as there are directrices, and therefore the differential equation, characterizing the surface independent of the directrices, and therefore not involving the functions, will be of an elevated order, generally exceeding the number of functions. These principles we shall now illustrate, confining ourselves, however, to the cases where the generating curve is a straight line.

174. Surfaces generated by a straight line are divisible into two classes, each of which has distinct properties; viz. into *twisted* surfaces, where two consecutive positions of the generating line are never in the same plane; and into *developable* surfaces, where two consecutive positions of the generating line are always in the same plane.

One remarkable difference in the properties of these two classes has reference to their contact with the tangent plane; in both, the tangent plane at any point (since it contains the tangent lines to all curves traced on the surface through that point, and a straight line is its own tangent) must contain the generating line passing through that point; but

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in twisted surfaces the tangent planes along a generating line are all distinct, and each touches the surface only in a point; whereas in developable surfaces, it is one and the same plane which touches the surface at all points along a generating line. For let DC, $D'C'$ (fig. 54.) be consecutive positions of the generating line of a twisted surface, BC, ED sections of the surface through any points D and C, and TC, AD tangent lines to those sections. Then the tangent planes at all points along DC contain DC; and at D and C the tangent planes are ADC, TCD, which cannot be coincident unless AD, TC are in the same plane; but in that case $D'C'$ (since between the positions DC, $D'C'$, the generating line may be supposed to glide along AD, CT) would be in the same plane with DC, which is impossible, since the surface is twisted. But if the surface be developable, then since $D'C'$ is in the same plane with DC, the tangents AD, TC are in the same plane, and therefore the tangent planes at D and C, as well as at every other point along DC, are coincident. This property has already been proved for cylinders and cones, which are evidently particular cases of developable surfaces.

175. We shall first consider twisted surfaces; and among the various ways in which the motion of a straight line may be governed so as to generate a twisted surface, we shall select the two following: viz. when the moveable line constantly passes through two fixed curves and remains parallel to a fixed plane, and when it constantly passes through three fixed curves.

176. To find the equation to the surface generated by a straight line which constantly passes through two given curves, and remains parallel to a fixed plane.

The motion of the generating line is here, manifestly, completely determined, as we have only to cut the curves by planes parallel to the fixed plane, and to join the points of section by straight lines, to obtain as many positions of the generating line as we please; and as the tangents to

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the fixed curves at those points of section will not usually be in the same plane, the surface will generally be twisted.

Taking the directing plane for that of xy, let the equations to the generating line in any position be

$$z = \alpha, \quad y = \beta x + \gamma,$$

and let the equations to the two directrices be

$$y = \phi_1(x), \quad z = \psi_1(x), \quad y = \phi_2(x), \quad z = \psi_2(x);$$

then successively combining the equations to the moveable line with those to each of the directrices, so as to eliminate x, y, z, we find

$$F_1(\alpha, \beta, \gamma) = 0, \quad F_2(\alpha, \beta, \gamma) = 0;$$

which we may suppose reduced to the forms

$$\beta = \phi(\alpha), \quad \gamma = \psi(\alpha).$$

Hence, eliminating α, β, γ between these and the equations to the generating line, we have for the equation to the surface

$$y = x\phi(z) + \psi(z).$$

Cor. To find the differential equation, we have, differentiating successively with respect to x and y, and dividing one result by the other,

$$0 = \phi(z) + x\phi'(z)p + \psi'(z)p,$$

$$1 = x\phi'(z)q + \psi'(z)q,$$

$$\frac{p}{q} = -\phi(z);$$

therefore, differentiating again, and dividing one result by the other, employing the usual notation,

$$\frac{qr - ps}{q^2} = -\phi'(z)p, \quad \frac{qs - pt}{q^2} = -\phi'(z)q;$$

$$\therefore \; q^2r - 2pqs + p^2t = 0,$$

the equation common to all surfaces of this class.

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177. To find the equation to the surface generated by a straight line subject to constantly pass through three given fixed curves.

It is easily seen that the motion of the generating line will be completely determined; for conceive any point in one of the fixed curves to be the common vertex of two conical surfaces, having the other two fixed curves for their directrices; then these surfaces will intersect one another in a finite number of straight lines, each of which passes through the three fixed curves, and is therefore a position of the generating line; and as we take fresh points for the vertex, the successive generating lines belonging to the same sheet, will pass through points contiguous to one another on the three directrices. Also the surface will generally be twisted, because the tangents to the three directrices at the points where a generating line meets them, will not, except in very particular cases, be in the same plane.

Let the equations to the generating line be

$$x = az + \gamma, \quad y = \beta z + \delta,$$

containing four variable parameters, a, β, γ, δ; and, as explained in Art. 173, let the three relations among the parameters necessary for the line passing through the three given directrices, be obtained and reduced to the form

$$\beta = \phi(a), \quad \gamma = \psi(a), \quad \delta = \pi(a);$$

therefore, substituting for β, γ, δ in the equations to the generating line, we have

$$x = az + \psi(a), \quad y = z\phi(a) + \pi(a);$$

and it remains to eliminate a between these equations, in order to get the equation to the surface; but this cannot be done without particularizing the functions, or the directrices on which they depend. Therefore we must retain the above system of equations to represent this family of surfaces, regarding a as an indeterminate quantity to be eliminated, after the forms of the functions in each indi-

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vidual case have been determined. It may be observed that the various other modes of generating this sort of surface, as for instance when we replace one or more of the directrices by a surface which the moveable line is always to touch, are reducible to this method; for in every case we may take for the three directrices, any three sections of the generated surface, made at pleasure.

Cor. To obtain the differential equation independent of the directrices, we must, as before, obtain by successive differentiations of the above system, a sufficient number of equations to eliminate α, and the functions ϕ, ψ, π, and their derived functions; the result is a complicated equation of the third order.

When in the two preceding cases, the directrices all become straight lines, the surfaces generated become respectively the hyperbolic paraboloid, and the hyperboloid of one sheet, as we have already seen; they are the only twisted surfaces whose equations do not rise above the second degree.

178. We next come to the consideration of the second class of surfaces which admit of being generated by a straight line, the characteristic law of the motion being that two consecutive positions are always in the same plane. Before proceeding to point out the various modes in which this condition may be satisfied, we shall shew that surfaces generated in this manner are *developable*; that is, supposing them flexible but inextensible, they may, without rumpling or tearing, be made to coincide with a plane in all their points. Let fig. 57. represent a surface of this sort, and let AN, $A'N'$, $A''N''$, &c. be positions of the generating line indefinitely near to one another; then from the definition of the surface, AN will be intersected by $A'N'$ in some point m, $A'N'$ by $A''N''$ in m', and the latter by the next generating line in m'', and so on; so that these successive points of intersection will form a polygon $mm'm''\ldots$, or rather a continuous curve to which all the generating lines are tangents; also AN and $A'N'$,

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and similarly every pair of consecutive generating lines, will include a sectorial area AmA' of indefinite length, but infinitely small angle, which may be regarded as a plane element of the surface. If now the first of these elements be turned about its line of intersection with the second, till they are in the same plane; and then the system formed by these two be turned about the line of intersection of the second and third till they are in the same plane; and if this operation be continued through all the plane elements, they will all thus be brought into one plane, and the given surface will be developed without rumpling or tearing.

179. We have already seen that the plane which touches a developable surface in any point M is the tangent plane at every point in the generating line passing through M; and that to construct the tangent plane to any point M, we have only to draw a tangent line MT to any curve on the surface passing through that point, then TMN is the tangent plane required.

180. As the generating lines are all tangents to the curve $mm'm''\ldots$ formed by their perpetual intersection, the surface may be supposed to be generated by a moveable straight line which is always a tangent to a fixed curve; the curve must of course be of double curvature, otherwise the surface generated would be a plane. Hence it is sufficient to assign *one* fixed directrix (to which the generating line must be always a tangent) to completely determine a developable surface. If the equations to the fixed curve be

$$x = \phi(z), \quad y = \psi(z),$$

the equation to the surface will result from eliminating a from the equations to the line touching the curve at a point for which $z = a$, which (Art. 123.) are

$$x - \phi(a) = \phi'(a)(z - a), \quad y - \psi(a) = \psi'(a)(z - a);$$

the elimination cannot be effected without assigning the forms of the functions; but every developable surface may

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be represented by the above system of equations, regarding α in the first as a function of y and z to be determined from the second.

181. The curve $mm'm''\ldots$ is called the *edge of regression* of the surface which its tangent generates; the reason of which is that the two portions of any tangent, produced both ways from the point of contact, generate two distinct sheets of the surface which are united in the curve; so that if we wished to pass from a point in one sheet to a point in the other without quitting the surface, our path would have a point of regression at the curve, provided the two points were not in the same generating line. In conical surfaces the edge of regression is reduced to a point, in cylindrical surfaces it is removed to an infinite distance.

182. We may also generate a developable surface, by assigning *two* fixed directrices through which the moveable line is always to pass; in that case the points in which the moveable line meets them must be constantly so chosen that the tangent lines to the directrices at those points are in the same plane; then, as the moveable line in passing into its consecutive position may be supposed to glide along those tangents, every two consecutive positions will be in the same plane. Consequently, after having obtained two relations among the parameters which enter into the equations to the moveable line, by subjecting it to meet the two directrices, we shall obtain a third by expressing that the tangents to the directrices at the points where the moveable line meets them, are in the same plane; hence only one arbitrary parameter will remain in the equations, by eliminating which we shall obtain the equation to the surface.

183. There is yet another way of expressing the generation of developable surfaces; for since every two consecutive generating lines include between them a plane sectorial area of indefinite length, we may regard these elements as form-

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ing parts of the successive positions of a plane subject to move according to a given law. This method will usually be found the most convenient in practical applications; and it leads easily to the general differential equation to developable surfaces, and to other results, which we shall now obtain by means of it.

184. To find the general equation to developable surfaces, considered as generated by the consecutive intersections of the positions assumed by a plane subject to move after a given law.

The law according to which the plane moves will vary with each particular surface; but in order that the motion may be completely determined, and that a single surface and not an infinite number of surfaces may be generated, it must leave only one arbitrary parameter in the equation to the moveable plane. The successive positions assumed by the plane by virtue of the infinitely small variations of the parameter, will cut one another consecutively in straight lines, which taken two and two are in the same plane; these lines will therefore form a developable surface touched by all the planes. Hence the law of succession of the planes which generate a developable surface requires that two of the three arbitrary constants which enter into the equation to a plane should be functions of the third, or, which is the same thing, that they should all be functions of the same parameter α; let therefore

$$z = x\phi(\alpha) + yf(\alpha) + \psi(\alpha)$$

be the equation to the generating plane in one of its positions depending upon the parameter α; then the equation to the plane, which differs insensibly from it in position, will be

$$z = x\{\phi(\alpha) + \phi'(\alpha).\delta\alpha + \&c.\} + y\{f(\alpha) + f'(\alpha).\delta\alpha + \&c.\}$$
$$+ \psi(\alpha) + \psi'(\alpha).\delta\alpha + \&c.;$$

and the co-ordinates of the points in which they intersect must satisfy these two equations; that is, they must satisfy

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$$z = x\phi(a) + yf(a) + \psi(a),$$

$$0 = x\{\phi'(a) + \&c.\} + y\{f'(a) + \&c.\} + \psi'(a) + \&c.;$$

or if the planes be consecutive, making $\delta a = 0$, the equations to their line of intersection are

$$z = x\phi(a) + yf(a) + \psi(a) \ldots\ldots (1),$$

$$0 = x\phi'(a) + yf'(a) + \psi'(a) \ldots\ldots (2);$$

and the general equation to developable surfaces will result from eliminating a between these equations; but as the elimination cannot be effected without fixing the forms of the functions ϕ, f, ψ, that is, without particularizing the surface, we must retain the above system to represent this family of surfaces, regarding a in the former as a function of x and y to be determined by the latter.

Cor. Hence the differential equation to developable surfaces can be obtained; for, differentiating (1) successively with respect to x and y, regarding a as a function of x and y to be determined from (2), we have

$$p = \phi(a) + \{x\phi'(a) + yf'(a) + \psi'(a)\} d_x a,$$

$$q = f(a) + \{x\phi'(a) + yf'(a) + \psi'(a)\} d_y a,$$

which by virtue of equation (2) are reduced to

$$p = \phi(a), \quad q = f(a);$$

and since p and q are functions of the same quantity, they are functions of one another,

$$\therefore p = \pi(q),$$

which is the differential equation of the first order to developable surfaces, containing one arbitrary function; this may be made to disappear by differentiating the equation $p = \pi(q)$ successively with respect to x and y, and dividing one result by the other; for we have

$$r = \pi'(q) \cdot s, \quad s = \pi'(q) \cdot t;$$

$$\therefore rt - s^2 = 0,$$

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the differential equation of the second order common to all developable surfaces.

185. To find the equations to the edge of regression of a developable surface.

The edge of regression is the locus of the intersections of the successive generating lines; and to find its equations we must combine the equations to any generating line (1) and (2), in which α is an arbitrary constant, with the equations to the consecutive generating line resulting from (1) and (2), by changing α into $\alpha + \delta\alpha$; we shall thus obtain four equations which by reduction are equivalent to the three

$$z = x\phi(\alpha) + yf(\alpha) + \psi(\alpha),$$
$$0 = x\phi'(\alpha) + yf'(\alpha) + \psi'(\alpha),$$
$$0 = x\phi''(\alpha) + yf''(\alpha) + \psi''(\alpha);$$

these, for any value of α, will furnish the three co-ordinates x, y, z of the point where the generating line corresponding to that value of α, is intersected by the consecutive generating line; so that in eliminating α from the above system, we shall have the equations to the locus of all such points of intersection; that is, to the edge of regression of the surface, by the tangent line to which the surface may be considered as generated.

186. To find the equation to the developable surface which touches at the same time two given curve surfaces.

Let the equations to the two surfaces be

$$F_1(x_1, y_1, z_1) = 0, \quad F_2(x_2, y_2, z_2) = 0,$$

and let $p_1 q_1$, $p_2 q_2$, denote the values of $d_x z$ and $d_y z$ for the first and second surface respectively. Then the moveable plane, since it touches the first surface, will have for its equation

$$z - z_1 = p_1(x - x_1) + q_1(y - y_1);$$

and in order that it may touch the second surface, that is, in order that it may coincide with the tangent plane at the unknown point $x_2 y_2 z_2$, we must join the conditions

$$p_1 = p_2, \quad q_1 = q_2, \quad z_1 - p_1 x_1 - q_1 y_1 = z_2 - p_2 x_2 - q_2 y_2.$$

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If therefore from these three equations, and the two equations to the surfaces, we determine the five unknown quantities y_1, z_1, x_2, y_2, z_2 in terms of x_1, the equation to the moveable plane will take the form

$$z = x\phi(x_1) + yf(x_1) + \psi(x_1);$$

and by eliminating the arbitrary quantity x_1 between this equation and the derived equation with respect to x_1, we shall obtain the equation to the required developable surface.

If we suppose an opaque body to be illuminated by a luminous one, the surfaces which circumscribe the umbra and penumbra occasioned by the interposition of the opaque body, are two sheets of the developable surface which touches both the bodies; and its lines of contact with the surfaces of the bodies are, one, the curve on the opaque body which separates the illumined and obscure parts; and the other, the curve on the luminous body which separates the illuminating part from that which can send no rays to the other body. Consequently this problem is the general problem of umbras and penumbras.

Similarly, we may find the developable surface generated by the intersections of the tangent planes to a surface along an assigned curve traced on the surface.

187. We may arrive at the differential equation to developable surfaces, and at the equation to the generating line passing through any point, by the following method.

Since a tangent plane applied to the surface at any point, touches it in a series of points the projection of which on any of the co-ordinate planes is a straight line, and since the equation to the tangent plane at a point xyz is

$$z' = px' + qy' + z - px - qy,$$

if $y' = mx' + n$ be the equation to the projection of the generating line which passes through the point xyz, and in the values of p, q, and $z - px - qy$, we change y into

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$mx + n$, the above equation must remain the same for all values of x whatever, and therefore the quantities p, q, $z - px - qy$, which become functions of x only, must have their differential coefficients with respect to x nothing,

$$\therefore\ d_x p = r + s d_x y = 0,$$

$$d_x q = s + t d_x y = 0,$$

$$d_x (z - px - qy) = p + q d_x y - (p + q d_x y) - x d_x p - y d_x q = 0.$$

Hence, eliminating $d_x y$, $rt - s^2 = 0$, the equation required; and $y' - y = -\frac{r}{s}(x' - x) = -\frac{s}{t}(x' - x)$ is the equation to the projection of the generating line which passes through the point $x\,y\,z$.

188. When a developable surface is made plane, the absolute lengths of any determinate portions of the generating lines, as well as of any curve traced on the surface, are not altered; and the angle which the tangent line to the curve at any point forms with the generating line through that point also remains unaltered, but the curvature at any point is changed.

For let AN, $A'N'$, &c. (fig. 57.) be consecutive positions of the generating line of a developable surface, and let PR, $P'R'$, &c. be the chords produced of a curve traced on this surface, and meeting the generating lines in the points P, P', &c.; and for the surface let us substitute the series of plane sectorial elements AmA', $A'm'A''$, &c., and for the curve the polygon $PP'P''$, &c. Then in turning the first plane element about $A'm$, to bring it into the same plane with the second, and in turning the system formed by the two about $A''m'$ to bring it into the same plane with the third, and so on, it is evident that neither the length of PN, $P'N'$, &c., nor the angles which they form with PP', $P'P''$, &c., nor the lengths of the latter are altered, but that the angles $RP'R'$, $R'P''R''$, &c. will be changed; and as this continues true, however small we take the plane

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elements, the properties announced above for the surface and the curve traced upon it, are established.

189. If the polygon be such that, upon bringing all the plane elements into the same plane, it becomes a straight line, then two consecutive sides must always make the same angle with the intermediate generating line; that is, for every point we must have

$$\angle m'P'R = m'P'R';$$

for when this condition is fulfilled, in the developed surface every two consecutive sides will be the prolongment of one another. Also since RP', $R'P'$, make equal angles with $P'N'$, they may be regarded as two generating lines of a conical surface whose axis is $P'N'$, and therefore the plane $RP'R'$ will ultimately be the tangent plane to this surface along $P'R$, and therefore perpendicular to $RP'N'$, which is a meridian plane of the cone containing $P'R$; hence, passing to the curve, the plane $RP'R'$ is its osculating plane at P', and $RP'N'$ is the tangent plane to the surface at the same point, and these planes are perpendicular to one another; also since, when the surface is developed, the curve becomes a straight line, it is the shortest line which can connect two points on the surface through which it passes; hence the shortest line on a developable surface has the property that its osculating plane at every point is perpendicular to the plane touching the surface at that point, or contains the normal to the surface at that point.

190. That the same property is true for the shortest line traced on any surface whatever, appears thus. If on any proposed surface we conceive the shortest line between any two points to be traced, we may describe a developable surface which shall touch the surface according to that curve; consequently the curve of contact will be the shortest line between the same points on the developable surface, and therefore its osculating plane at every point must contain the normal to the developable surface, *i. e.* the normal

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to the proposed surface, at that point. Hence if $y = \phi(x)$, $z = \psi(x)$, be the unknown equations to the shortest line on a surface for which $d_x z = p$, $d_y z = q$; expressing that the osculating plane at any point contains the normal to the surface at that point, we find

$$d_x^2 y (1 + p d_x z) = d_x^2 z (p d_x y - q);$$

which together with $d_x z = p + q d_x y$ will determine the curve. If we take (s) the length of the arc for independent variable, the equations to the curve will assume the symmetrical forms

$$d_s^2 x + p d_s^2 z = 0, \quad d_s^2 y + q d_s^2 z = 0.$$

191. From the property that the lengths of the generating lines of a developable surface, and of any curve traced upon it, as also the angles which the generating lines form with the curve, are not altered by the development of the surface, we can find the nature of the curve when the surface is made plane; or conversely, a curve being traced upon a plane, we can find its nature when the plane is applied to a given surface. This we shall now shew in the case of cylindrical and conical surfaces.

192. Case 1. Let the surface be cylindrical, and x, y, z the co-ordinates of any point in a curve traced upon it; then the equations to the generating line which passes through that point are

$$z' - z = m(x' - x), \quad y' - y = n(x' - x);$$

and the equations to the tangent line of the curve are

$$z' - z = d_x z (x' - x), \quad y' - y = d_x y (x' - x);$$

let i denote the angle between these two lines, γ the inclination of the generating line to the axis of x, and s the length of the curve;

$$\therefore \cos i = \frac{m d_x z + n d_x y + 1}{\sqrt{1+m^2+n^2}\sqrt{1+(d_x z)^2+(d_x y)^2}} = \frac{\cos\gamma}{d_x s}(m d_x z + n d_x y + 1).$$

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Now when the cylinder is developed, the generating lines preserve their parallelism, and may therefore be taken for the ordinates; let a line perpendicular to them be the axis of the abscissæ, and α, β, the co-ordinates of the point in question;

$$\therefore \cos i = \frac{1}{d_\beta s};$$

$$\therefore d_x\beta = \cos\gamma(m d_x x + n d_x y + 1).$$

Also

$$1 + (d_x y)^2 + (d_x x)^2 = (d_x\alpha)^2 + (d_x\beta)^2 = (d_x\alpha)^2\{1 + (d_\alpha\beta)^2\},$$

each being the value of $(d_x s)^2$; and if between these and the two equations to the curve we eliminate x, there will remain a differential equation between α and β which will be that to the required plane curve. But if the latter equation be given, that is, $\beta = f(\alpha)$, we may eliminate α between the two above equations, and there will remain a differential equation between x, y, and z, which, with the given equation to the surface, will be the equations to the required curve of double curvature.

Cor. If the curve on a plane be a straight line, the angle i is constant;

$$\therefore \cos\gamma(m d_x x + n d_x y + 1) = \cos i \,.\, d_x s$$

is the equation for determining its nature, when the plane is applied to a cylinder; the curve is called the Helix, and we have for its length, by integration,

$$\cos\gamma(mx + ny + x) = \cos i(s + C).$$

It is manifestly the shortest line which can join two points on the surface of the cylinder.

193. Case 2. Let the developable surface be a conical surface; a, b, c the co-ordinates of its vertex, x, y, z co-ordinates of a point in a curve traced upon it, and r the distance of that point from the vertex. When the cone is developed, the generating lines become radii vectores, and

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if θ be the angle which r forms with some fixed radius vector, equating the two values of $(d_x s)^2$, we have

$$1 + (d_x z)^2 + (d_x y)^2 = \{r^2 (d_r \theta)^2 + 1\} (d_x r)^2,$$

$$\text{and} \quad r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2;$$

from which, together with the two equations to the curve, we may eliminate x, y, z, and there will remain a relation between r and θ, the equation to the plane curve when the surface is developed; or, if $r = f(\theta)$, and the equation to the conical surface be given, we may eliminate r and θ, and so arrive at the remaining equation to the curve of double curvature resulting from the application of a plane on which a given curve is traced to a given conical surface.

Cor. All curves which make a constant angle with the generating line of the cone are characterized by the equation $d_s r = \cos i$, and become equiangular spirals when the surface is developed. Also all curves which become straight lines when the surface is developed, and which therefore are the shortest lines which can connect two points on its surface, are characterized by the equation $d_x s = d_x \sqrt{r^2 - a^2}$, a being the length of the perpendicular let fall upon the line from the vertex as origin, or

$$\sqrt{1 + (d_x z)^2 + (d_x y)^2} = d_x \sqrt{x^2 + y^2 + z^2 - a^2};$$

or finally, by developing this equation,

$$(x d_x y - y)^2 + (x d_x z - z)^2 + (y d_x z - z d_x y)^2 = a^2 \{1 + (d_x z)^2 + (d_x y)^2\}.$$

When the surface is a right cone on a circular base, the problem admits of a very simple solution, as will be seen in the examples appended to this Section.

194. Developable surfaces, considered as generated by the successive intersections of planes drawn after a given law, are only a particular case of a general family of surfaces to which we shall in the next place direct our attention.

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195. To find the equation to a surface which envelopes a series of surfaces described after a given law.

Let $u = f(x, y, z, a) = 0$ be the equation to a surface containing, besides other constants, a parameter a; then if we give a particular value to a, the form and position of the surface in space will be completely determined; and if we give to it all possible consecutive values, we shall obtain an infinite number of corresponding surfaces, usually intersecting one another two and two. The surface formed by these successive intersections, and having with each one of the surfaces the infinitely narrow zone in common which is contained between the curves in which that individual surface is intersected by the preceding and succeeding ones, (and which consequently envelopes each one of the first series of surfaces, supposing them all to exist together, and touches it according to a curve of intersection,) has been named by Monge the *Envelope;* and the curve in which any two consecutive surfaces intersect, the *Characteristic* of the Envelope, because it indicates the mode of its generation; thus the characteristic of developable surfaces, which are generated by the intersections of planes, is a straight line; and that of surfaces generated by the intersections of spheres, a circle. To find the equation to the envelope, we observe that if after having given the parameter a determined value a in $u = 0$, we give it a new value $a + \delta a$ differing insensibly from the former, we obtain the equation to a second surface differing in form and position insensibly from the first, and intersecting it in a series of points the co-ordinates of which satisfy the equations

$$u = 0, \qquad u + d_a u \,.\, \delta a + d_a^2 u \,.\, \frac{(\delta a)^2}{2} + \&c. = 0,$$

$$\text{or} \quad u = 0, \qquad d_a u + \&c. = 0;$$

or if the surfaces be consecutive,

$$u = 0, \qquad d_a u = 0;$$

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it follows therefore that the equations to the characteristic or curve of intersection of two consecutive surfaces, corresponding to the value a of the parameter, are

$$f(x, y, z, a) = 0, \qquad d_a f(x, y, z, a) = 0;$$

and since the envelope is formed by the assemblage of the characteristics, its equation will result from eliminating a between the same equations. As this elimination, however, cannot be effected without assigning the form of f, we must retain the above system of equations to represent the envelope, regarding a in the former as a function of x and y to be determined from the latter.

196. Again, if in the equations to the characteristic, after having given the parameter a particular value a, which determines the position of the characteristic in space, we give it a new and insensibly increased value $a + \delta a$, the two resulting equations will be those to a second characteristic, differing insensibly in form and position from the first, and intersecting it in general in a finite number of points, the co-ordinates of which satisfy the two sets of equations

$$u = 0, \qquad d_a u = 0,$$

$$u + d_a u \,.\, \delta a + \&c. = 0, \qquad d_a u + d_a^2 u \,.\, \delta a + \&c. = 0;$$

which latter two equations, by virtue of the former, are equivalent to

$$d_a u + \&c. = 0, \qquad d_a^2 u + \&c. = 0.$$

Consequently the co-ordinates of the points in which two consecutive characteristics intersect, satisfy the three equations

$$u = 0, \qquad d_a u = 0, \qquad d_a^2 u = 0;$$

and from these equations we may either determine x, y, and z in terms of a, that is, the co-ordinates of the point in which a given characteristic is intersected by the consecutive; or eliminate a between them, and so find the two

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equations in x, y, and z, to the curve formed by all the successive points of intersection. This curve will be touched by all the characteristics, and will be the edge of regression of the envelope; for, as explained in the case of developable surfaces, the portions of the characteristics which are on contrary sides of their points of contact with the curve under consideration, will form two distinct sheets of the envelope, and these sheets will touch one another in that curve and have it for their common limit.

Cor. The propriety of the term *Characteristic* will be more apparent, if we generalise the equation $u = 0$, and suppose it to contain, together with α, a function of α, $\phi(\alpha)$, so as to have the form

$$f\{x, y, z, \alpha, \phi(\alpha)\} = 0.$$

Then whatever form we give to ϕ, the intersection of two consecutive surfaces represented by this equation will always be a curve of the same sort, and will therefore offer a character common to the whole family of surfaces. For this curve of intersection will be represented by the equations (making $\phi(\alpha) = \beta$)

$$f(x, y, z, \alpha, \beta) = 0, \qquad d_{(\alpha)}f + d_{(\beta)}f \cdot d_\alpha\beta = 0;$$

and we see that by changing β, we shall only alter the constant parts of these equations, without at all affecting the way in which x, y, z enter into them; so that for all values of β, the curve represented by them will be of the same species.

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197. The following are examples of finding the equations to twisted and developable surfaces, and envelopes.

I. To find the equation to the twisted surface of which the directrices are two vertical circles, having the opposite sides of a horizontal parallelogram for diameters, and a straight line passing through the center of the parallelogram perpendicular to the planes of the circles.

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Let the center of the parallelogram be taken for origin and its plane for that of xy, and the rectilinear directrix for the axis of y; then the equations to the three directrices will be

$$x = 0 \qquad z = 0,$$
$$y = -b \qquad (x-a)^2 + z^2 = r^2,$$
$$y = +b \qquad (x+a)^2 + z^2 = r^2.$$

Also let the equations to the moveable line be

$$x = \alpha(y-\beta), \qquad z = \gamma(y-\beta),$$

so that it already fulfils the condition of meeting the axis of y, and one of the parameters consequently is eliminated; then, expressing that it passes through each of the circles, we have

$$\left.\begin{aligned}\{\alpha(b+\beta)+a\}^2 + \gamma^2(b+\beta)^2 &= r^2\\ \{\alpha(b-\beta)+a\}^2 + \gamma^2(b-\beta)^2 &= r^2\end{aligned}\right\}\ldots(1);$$

therefore, by subtraction, $\beta(b\alpha^2 + a\alpha + b\gamma^2) = 0$;

and rejecting the value $\beta = 0$ which would make the moveable line always pass through the origin and so generate an oblique cone, we have

$$\alpha^2 + \gamma^2 + \frac{a}{b}\alpha = 0\ldots(2),$$

by virtue of which, either of the equations (1) gives

$$a\alpha(b^2 - \beta^2) = b(r^2 - a^2)\ldots(3);$$

and it remains to eliminate α, β, γ between (2) and (3), and the equations to the generating line. Substituting the values of α and γ, given by the latter in (2), we find

$$\beta = y + \frac{b}{a}\frac{x^2+z^2}{x}, \qquad \text{and then } \alpha = -\frac{a}{b}\frac{x^2}{x^2+z^2};$$

therefore, substituting these values of α and β in (3), and reducing, we find for the equation to the surface

$$\{axy + b(x^2+z^2)\}^2 = b^2r^2x^2 + b^2z^2(r^2-a^2).$$

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II. To find the equation to the developable surface generated by a straight line which constantly touches a Helix.

The equations to the Helix are (p. 124.)

$$x = a\cos\frac{z}{na}, \qquad y = a\sin\frac{z}{na}.$$

Therefore at a point for which $z = a$, the equations to the tangent are

$$x - a\cos\frac{a}{na} = -\frac{1}{n}\sin\frac{a}{na}(z-a),$$

$$y - a\sin\frac{a}{na} = \frac{1}{n}\cos\frac{a}{na}(z-a);$$

$$\therefore\ x\cos\frac{a}{na} + y\sin\frac{a}{na} = a, \quad \text{and } \tan\frac{a}{na} = \frac{a\sqrt{x^2+y^2-a^2}-xy}{y^2-a^2};$$

also, adding the squares of the above equations, we find

$$x^2 + y^2 - a^2 = \frac{1}{n^2}(z-a)^2;$$

$$\therefore\ n\sqrt{x^2+y^2-a^2} = z - na\tan^{-1}\left(\frac{a\sqrt{x^2+y^2-a^2}-xy}{y^2-a^2}\right),$$

the equation to the surface.

III. If a series of planes, passing through a fixed point in the axis of z, have their traces on the plane of xy all of the same length; to find the equation to the developable surface formed by their intersections.

If c = distance of the fixed point from the origin, a = length of the trace on the plane of xy, and α = the angle which it makes with the axis of x, the equation to the corresponding plane is

$$\frac{x}{a\cos\alpha} + \frac{y}{a\sin\alpha} + \frac{z}{c} = 1;$$

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differentiate with respect to α,

$$\frac{x}{a}\frac{\sin\alpha}{(\cos\alpha)^2} - \frac{y}{a}\frac{\cos\alpha}{(\sin\alpha)^2} = 0; \quad \therefore (\tan\alpha)^3 = \frac{y}{x};$$

therefore, substituting in the equation to the plane which may be written

$$\frac{x}{a}\sqrt{1+(\tan\alpha)^2} + \frac{y}{a}\sqrt{1+(\cot\alpha)^2} + \frac{z}{c} = 1, \text{ we find}$$

$$\frac{x}{a}\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}} + \frac{y}{a}\sqrt{1+\left(\frac{x}{y}\right)^{\frac{2}{3}}} + \frac{z}{c} = 1,$$

$$\text{or } \frac{1}{a}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}} + \frac{z}{c} = 1, \text{ the required equation.}$$

The equations to the characteristic in this case are

$$\frac{x}{a(\cos\alpha)^3} + \frac{z}{c} = 1, \qquad \frac{y}{x} = (\tan\alpha)^3.$$

IV. A given curve being traced on the surface of a right cone whose base is a circle, to determine its equation when the surface is developed.

Let P (fig. 58.) be a point of the curve situated on the generating line CR, Q its projection on a plane perpendicular to the axis; then our object is to find a relation between $CP = r$, and the angle which CP makes with CA, when the cone is developed; call this angle θ, and let

$$CQ = r', \quad QCx = \theta', \quad m = \operatorname{cosec} ACD;$$

now, when the cone is developed, θ is subtended by a circular arc, length AR;

$$\therefore \theta . AC = \theta' . AD, \quad \text{or, } \theta' = m\theta;$$

$$\text{also } CP . \sin PCD = CQ, \quad \text{or, } r' = \frac{r}{m};$$

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if, therefore, $f(r', \theta') = 0$ be the given equation to the locus of Q, $f\left(\frac{r}{m}, m\theta\right) = 0$ is the equation to the curve, when the surface of the cone is developed.

Conversely, if $f(r, \theta) = 0$ be the equation to a curve traced in a circular sector, and the sector be formed into a right cone, the equation to the projection of the curve on a plane perpendicular to the axis of the cone will be

$$f\left(mr', \frac{\theta'}{m}\right) = 0.$$

Ex. 1. A right cone being intersected by a sphere whose center is in its surface, to determine the equation to the curve of intersection when the cone is developed.

Let B in the plane of zx be the center of the sphere (fig. 58.) $BC = c$, $a =$ its radius, $\alpha = \frac{1}{2}$ vertical $\angle$ of the cone, then

$$(c \sin \alpha - x)^2 + (c \cos \alpha - z)^2 + y^2 = a^2$$

is the equation to the sphere, and $z^2 (\tan \alpha)^2 = x^2 + y^2$ is the equation to the cone; therefore, the equation to the projection of their intersection on the plane of xy is

$$c^2 + (x^2 + y^2)\{1 + (\cot \alpha)^2\} - 2cx \sin \alpha - 2c \frac{(\cos \alpha)^2}{\sin \alpha} \sqrt{x^2 + y^2} = a^2,$$

and its polar equation is, making $x = r' \cos \theta'$, $y = r' \sin \theta'$,

$$c^2 + r'^2 (\operatorname{cosec} \alpha)^2 - 2cr' \sin \alpha \cos \theta' - 2cr' \operatorname{cosec} \alpha (\cos \alpha)^2 = a^2.$$

But $m = \operatorname{cosec} \alpha$, $r = r' \operatorname{cosec} \alpha$, $\theta' = m\theta$;

therefore, when the cone is developed, the equation to the curve of intersection is

$$r^2 - 2cr\{(\sin \alpha)^2 \cos m\theta + (\cos \alpha)^2\} + c^2 - a^2 = 0.$$

Ex. 2. If a semicircle be described upon the bounding radius of a quadrant, to find the equation to its projection upon a plane perpendicular to the axis, when the quadrant is formed into a cone.

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Here $m = 2\pi \div \frac{\pi}{2} = 4$, also the equation to the semicircle is $r = a\cos\theta$; therefore the equation to its projection is

$$r' = \frac{a}{4}\cos\frac{\theta'}{4}.$$

V. The following are examples of Envelopes.

Ex. 1. To find the equation to the envelope of all right cones of a constant volume, whose axes are in the same straight line, and bases in the same plane.

The equation to the surface of any cone is

$$z = a\left(1 - \frac{\sqrt{x^2+y^2}}{b}\right),$$

where a = altitude, b = radius of base, and the origin is in the center of the base.

Let the given volume be that of a hemisphere, diameter $3c$;

$$\therefore \frac{\pi a b^2}{3} = \frac{\pi(3c)^3}{12}, \text{ or } a = \frac{27c^3}{4b^2}; \quad \therefore z = \frac{27c^3}{4b^2}\left(1 - \frac{\sqrt{x^2+y^2}}{b}\right);$$

differentiate with respect to the parameter b,

$$\therefore 0 = -\frac{27c^3}{2b^3} + \frac{81c^3}{4b^4}\sqrt{x^2+y^2}, \quad \text{or } b = \frac{3}{2}\sqrt{x^2+y^2};$$

$$\therefore z = \frac{27c^3}{9(x^2+y^2)}\left(1 - \frac{2}{3}\right) = \frac{c^3}{x^2+y^2}, \text{ the equation required.}$$

$$\text{Also } \sqrt{x^2+y^2} = \frac{2b}{3}, \quad \text{and } z = \frac{27c^3}{4b^2}\left(1 - \frac{2}{3}\right) = \frac{9c^3}{4b^2},$$

are the equations to the characteristic, that is, determine the radius, and position of the center of the circle, in which the cone the radius of whose base = b, is intersected by the consecutive cone of the same volume.

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Ex. 2. If the center of a sphere whose radius $= a$, describe a curve in the plane of xy, to find the equation to the *annular* surface which envelopes the sphere in every position.

Let α and β be the co-ordinates of the center of the sphere in any position, and $y = f(x)$ the equation to the curve which it describes, therefore $\beta = f(\alpha)$;

$$\therefore (x-\alpha)^2 + \{y - f(\alpha)\}^2 + z^2 = a^2$$

is the equation to the surface, and by differentiating with respect to α,

$$x - \alpha + \{y - f(\alpha)\} f'(\alpha) = 0;$$

these are the two equations to the characteristic, whose position and magnitude depend upon α; and if we eliminate α between them, we find the equation to the envelope. The characteristic in this case is manifestly a circle of constant radius whose center is a point in the curve which forms the axis of the surface, and whose plane is perpendicular to the tangent line at that point, as is expressed by the above equations; hence its nature is entirely independent of the curve whose equation is $y = f(x)$; it is also the curve of greatest inclination, for its tangent line at every point is perpendicular to the intersection, with the plane of xy, of the plane touching the surface at the same point, therefore its equation is (p. 127.)

$$d_x y = \frac{q}{p}.$$

The remaining equation is that to the envelope, since the characteristic is situated upon it, which we obtain by observing that the normal always meets the plane of xy in the curve which forms the axis of the surface, and the length is equal to the radius of the generating sphere;

$$\therefore -z\sqrt{1 + p^2 + q^2} = a, \quad \text{(Art. 110.)}$$

$$\text{or } z^2(1 + p^2 + q^2) = a^2;$$

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this might also have been obtained by eliminating the arbitrary function from the two equations to the characteristic; hence the integral of the equation $x^2(1+p^2+q^2)=a^2$ is represented by the system of equations to the characteristic.

Ex. 3. If the vertex of a right cone describe a given curve in the plane of xy, and its axis be always perpendicular to that plane, to find the equation to the surface which touches and envelopes it in every position.

Let α and β be the co-ordinates of the vertex, $y=f(x)$ the equation to the curve which it describes, therefore $\beta=f(\alpha)$; a = tangent of the angle which the side of the cone makes with the plane of xy;

$$\therefore\ (x-\alpha)^2+\{y-f(\alpha)\}^2=\frac{x^2}{a^2}$$

is the equation to the surface; differentiate twice successively with respect to α;

$$\therefore\ \{y-f(\alpha)\}f'(\alpha)+x-\alpha=0,\quad \{y-f(\alpha)\}f''(\alpha)-1-\{f'(\alpha)\}^2=0;$$

from which equations we may deduce as above the equation to the envelope, and the equations to its characteristic, and edge of regression.

The tangent plane to the envelope at any point, is also the tangent plane to the generating cone, and therefore makes a constant angle (γ) with the plane of xy; but

$$\cos\gamma=\frac{1}{\sqrt{1+p^2+q^2}};$$

therefore, since $a=\tan\gamma$, the differential equation to the surface is

$$p^2+q^2=a^2;$$

the integral of which is of course represented by the system of equations to the characteristic.

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Hence, if S denote the area of any portion of the surface;

$$d_x d_y S = \sqrt{1 + p^2 + q^2} = \sec \gamma, \quad \text{or} \quad S = A_x \sec \gamma;$$

that is, any portion of the surface bears a constant ratio to its projection on the plane of xy. The characteristic in this case is evidently a side of the generating cone, and therefore perpendicular to the intersection of the tangent plane with the plane of xy; therefore its two equations are

$$d_x y = \frac{q}{p}, \quad p^2 + q^2 = a^2;$$

hence the curve of greatest inclination of all surfaces generated in this manner is a straight line, inclined at a constant angle to the plane of xy.

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SECTION VIII.

ON THE CURVATURES OF CURVES IN SPACE.

198. PREPARATORY to finding the radius of curvature, and evolutes of a curve in space, consider figure (59), where for the curve is substituted an equilateral polygon $mm'm''\ldots$, and through the middle points of its sides are drawn planes respectively perpendicular to them, which intersect, two and two, in the lines kh, $k'h'$, $k''h''$, &c. Then the plane which contains the two consecutive sides mm', $m'm''$, is perpendicular to each of the planes gh, $g'h'$, and therefore to their common intersection kh; let kh meet this plane in the point q, then q is equidistant from the three angles m, m', m'', which is also true of every point in the line kh. The lines kh, $k'h'$, &c. will be parallel only when the sides of the polygon $mm'm''\ldots$ are in the same plane; in other cases, if they be produced till each meets its consecutive, they will form a polygon $hop\ldots$, the angular points of which are equidistant from four consecutive angles of the first polygon $mm'm''\ldots$. The point o for instance, since it is situated in kh, is equidistant from m, m', m''; and again, being situated in $k'h'$, it is equidistant from m', m'', m'''; that is, it is the center of a sphere passing through four consecutive angles, m, m', m'', m'''.

This being true when the number of sides of the polygon is indefinitely increased, it follows that the normal planes of a curve generate, by their perpetual intersection, a curve surface; also, since of the lines of intersection kh, $k'h'$, &c. every two consecutive ones are in the same plane, the surface which they generate is developable.

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199. On the same supposition, the polygon $hop\ldots$ becomes a curve, to which all the lines ko, $k'p$, &c. are tangents, that is, $hop\ldots$ is constantly touched by the straight line which generates the developable surface, and is therefore the limit to the surface, for no part of the surface can fall within the space towards which the curve is concave; moreover its tangent, being produced both ways from the point of contact, generates two sheets of the surface which are united and terminated in the curve; as was explained in Art (181), it is called *the edge of regression* of the developable surface, from the analogy it bears to a point of regression in a plane curve. This curve is also the locus of the points of intersection of three consecutive normal planes, or of the centers of spheres passing through four consecutive points of the first curve $mm'm''\ldots$, that is, it is the locus of the center of *spherical curvature*.

200. The plane gqg' passing through two consecutive tangents to the curve, and which is perpendicular to the intersection of two consecutive normal planes, is the osculating plane, whose equation was found in Art. 127; and the point q, in which that intersection meets it, is called the absolute center of curvature.

201. To find the equation to the surface generated by the perpetual intersections of the normal planes to a curve in space.

Let $y = \phi(x)$, $z = \psi(x)$, be the equations to the curve, and let $d_x y$, $d_x^2 y$, &c. be denoted by y', y'', &c.; then we have seen Art. (130), that the equation to the normal plane

$$x_{,} - x + (y_{,} - y)\, y' + (z_{,} - z)\, z' = 0, \quad (1),$$

and the derived equation with respect to x, viz.

$$(y_{,} - y)\, y'' + (z_{,} - z)\, z'' - 1 - y'^2 - z'^2 = 0, \quad (2),$$

are the equations to the line of intersection of two consecutive normal planes; $x_{,}$, $y_{,}$, $z_{,}$, denoting the co-ordinates of any point in that intersection. If therefore we eliminate x between them, we shall have the equation to the surface

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generated by the line of intersection in all its successive positions. On this surface lie all the evolutes of the curve, as will be shewn; it will be a cylinder if the given curve lie all in one plane, and a cone, with its vertex in the center, if the given curve be traced on the surface of a sphere.

202. To find the radius, and co-ordinates of the center, of spherical curvature of a curve of double curvature at any point.

The center of spherical curvature, or of the sphere which passes through four consecutive points of a curve, is the point of intersection of two consecutive generating lines of the surface considered in the last article; hence we must join to equations (1) and (2) of last article, the derived equations with respect to x, as explained Art. (185).

If therefore $x_{,}$, $y_{,}$, $z_{,}$ be the co-ordinates of the center of spherical curvature, we have in order to determine them, the equations,

$$x_{,} - x + (y_{,} - y)\, y' + (z_{,} - z)\, z' = 0,$$
$$(y_{,} - y)\, y'' + (z_{,} - z)\, z'' - (1 + y'^2 + z'^2) = 0,$$
$$(y_{,} - y)\, y''' + (z_{,} - z)\, z''' - 3\,(y'y'' + z'z'') = 0,$$

and the radius of the sphere

$$= \sqrt{(x_{,} - x)^2 + (y_{,} - y)^2 + (z_{,} - z)^2}.$$

Moreover if from these three equations we eliminate x, there will result two equations between $x_{,}$, $y_{,}$, $z_{,}$, which will be those to the locus of the center of spherical curvature, or to the edge of regression of the developable surface generated by the intersections of the normal planes.

203. To find the radius, and co-ordinates of the center of the absolute circle of curvature.

The absolute center of curvature is the point where the intersection of two consecutive normal planes meets the osculating plane.

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$$\text{Now if } \frac{z''y' - z'y''}{y''} = a, \quad \frac{z''}{y''} = b;$$

$$\text{and } \therefore a + z' = by'; \quad \text{also } 1 + z'^2 + y'^2 = a,$$

the equation to the osculating plane is

$$z_{,} - z = -a(x_{,} - x) + b(y_{,} - y);$$

and the equations to the line of intersection of two consecutive normal planes are

$$x_{,} - x = a(z_{,} - z) - \frac{ay'}{y''}, \quad y_{,} - y = -b(z_{,} - z) + \frac{a}{y''}; \quad \text{(Art. 130.)};$$

therefore at the point where they intersect,

$$z_{,} - z = -a^2(z_{,} - z) + \frac{aay'}{y''} - b^2(z_{,} - z) + \frac{ba}{y''};$$

$$\therefore z_{,} - z = \frac{a}{y''} \cdot \frac{ay' + b}{1 + a^2 + b^2} \ldots\ldots\ldots\ldots (1);$$

$$\text{similarly } x_{,} - x = -\frac{a}{y''} \cdot \frac{y' + bz'}{1 + a^2 + b^2} \ldots\ldots\ldots (2),$$

$$y_{,} - y = -\frac{a}{y''} \cdot \frac{az' - 1}{1 + a^2 + b^2} \ldots\ldots\ldots\ldots (3);$$

$$\therefore R^2 = \left(\frac{a}{y''}\right)^2 \cdot \frac{(ay' + b)^2 + (bz' + y')^2 + (az' - 1)^2}{(1 + a^2 + b^2)^2}$$

$$= \left(\frac{a}{y''}\right)^2 \cdot \frac{(a^2+1)y'^2 + (a^2+b^2)z'^2 + 1 + b^2 + 2(aby' + by'z' - az')}{(1 + a^2 + b^2)^2}.$$

$$\text{But } a + z' - by' = 0;$$

$$\therefore 2(aby' + by'z' - az') = a^2 + z'^2 + b^2y'^2;$$

$$\therefore R^2 = \left(\frac{a}{y''}\right)^2 \cdot \frac{y'^2 + z'^2 + 1}{1 + a^2 + b^2} = \frac{(1 + y'^2 + z'^2)^3}{y''^2 + z''^2 + (z''y' - z'y'')^2},$$

$$\text{or } R = \frac{(1 + y'^2 + z'^2)^{\frac{3}{2}}}{\sqrt{y''^2 + z''^2 + (z''y' - z'y'')^2}};$$

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this is the value of the absolute radius of curvature, and the co-ordinates of its center are given by equations (1), (2), and (3).

Cor. If we make s, the length of the arc, the independent variable instead of x, we find by the common process of changing the independent variable,

$$R = \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}},$$

where x'' denotes $d_s^2 x$, &c.; also the expressions for the co-ordinates of the center assume the following simple forms,

$$x_1 - x = \frac{x''}{x''^2 + y''^2 + z''^2}, \quad y_1 - y = \frac{y''}{x''^2 + y''^2 + z''^2},$$

$$z_1 - z = \frac{z''}{x''^2 + y''^2 + z''^2}.$$

204. We may however arrive at the expression for R by the following independent method, which it is useful to know.

Let P be a point of the curve (fig. 60.), and let the plane of the paper OPT' be the osculating plane at P, that is, the ultimate position of a plane drawn through PT' parallel to $P'T'$ the tangent line at a consecutive point P', when P' moves up to P; then we may consider the indefinitely small arc $PP' = \delta s$ as lying in the plane OPT', and $P'T'$ as intersecting PT' in some point M; and if PO, $P'O$, be the intersections of the normal planes at P and P' with the plane of the paper, the ultimate position of O is the center of the circle of curvature at P.

Let the angle between two consecutive tangents $TMT' = POP'$, which is called the angle of contingence, be denoted by ϵ, then because PP' may be considered as ultimately coinciding with the arc of the circle of curvature at P, we have, if $PO = R$,

$$R \cdot \epsilon = PP' \text{ or } R = \text{limit of } \frac{\delta s}{\epsilon}.$$

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Let x, y, z be co-ordinates of P, and instead of taking two equations between these quantities to represent the curve, let us suppose, which is always allowable, each of them to be a function of a quantity t; and let $t+i$ be the value of t corresponding to the point P'; also let u, v, w denote the cosines of the angles which PT makes with the axes of x, y, z; then the values of these cosines relative to $P'T'$ will be

$$u + u' \cdot \iota + \tfrac{1}{2} u'' \cdot \iota^2, \text{ &c., (denoting } d_t u \text{ by } u');$$

$$\therefore \cos \epsilon = u\,(u + u' \cdot \iota + \tfrac{1}{2} u'' \cdot \iota^2)$$

$$+ v\,(v + v' \cdot \iota + \tfrac{1}{2} v'' \cdot \iota^2) + w\,(w + w' \cdot \iota + \tfrac{1}{2} w'' \cdot \iota^2).$$

But $u^2 + v^2 + w^2 = 1$,

$$\therefore uu' + vv' + ww' = 0,$$

$$uu'' + vv'' + ww'' = -\,(u'^2 + v'^2 + w'^2),$$

$$\text{also } \cos \epsilon = 1 - \tfrac{1}{2} \epsilon^2,$$

$$\therefore \epsilon^2 = \iota^2 \left\{(u')^2 + (v')^2 + (w')^2\right\}.$$

But (Art. 131.) $u = \dfrac{1}{d_s s} = \dfrac{x'}{s'}$, $\quad v = \dfrac{y'}{s'}$, $\quad w = \dfrac{z'}{s'}$,

$$\therefore u' = \frac{s'x'' - x's''}{(s')^2}, \text{ &c.}$$

$$\therefore \epsilon^2 = \frac{\iota^2}{(s')^4} \left\{(s'x'' - x's'')^2 + (s'y'' - y's'')^2 + (s'z'' - z's'')^2\right\},$$

and expanding each of the binomials, we have

$$\text{sum of first terms} = (s')^2 \left\{(x'')^2 + (y'')^2 + (z'')^2\right\},$$

$$\text{sum of second terms} = -\,2s's'' \left\{x'x'' + y'y'' + z'z''\right\} = -\,2(s's'')^2,$$

$$\text{sum of last terms} = (s'')^2 \left\{(x')^2 + (y')^2 + (z')^2\right\} = (s's'')^2;$$

$$\therefore \epsilon^2 = \left(\frac{\iota}{s'}\right)^2 \left\{(x'')^2 + (y'')^2 + (z'')^2 - (s'')^2\right\}.$$

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$$\text{But } R = \text{limit of } \frac{\delta s}{\epsilon} = \text{limit of } \frac{s' \cdot \iota}{\epsilon};$$

$$\therefore R = \frac{(s')^2}{\sqrt{(x'')^2 + (y'')^2 + (z'')^2 - (s'')^2}}.$$

This expression manifestly agrees with those already obtained for R, if we suppose the independent variable t to become equal to s or x.

205. A curve of double curvature may either have three consecutive elements in the same plane, or two consecutive elements in the same straight line; the first is called a *simple* inflexion, that is, when the curve at that point becomes plane, and therefore the radius of spherical curvature infinite; and the second a *double* inflexion, as necessarily including the former, that is, when the curve at that point becomes a straight line, and therefore the radius of absolute curvature infinite.

If in Art. 202, we actually determine the values of $x_{,} - x$, $y_{,} - y$, $z_{,} - z$, we find for their common denominator the expression

$$z''' y'' - y''' z'';$$

hence the condition $d_x^3 z\, d_x^2 y - d_x^3 y\, d_x^2 z = 0$, expressing that the center of spherical curvature is at an infinite distance, will determine the points of simple inflexion, and agrees with Art. 128.

At points of double inflexion, the absolute radius of curvature changes its sign, and the curve, after being concave in one direction, crosses its tangent line and becomes concave in the opposite direction; hence also its projections on the planes of zx and xy cross their tangents, and have corresponding points of contrary flexure; at such points therefore, and at cusps,

$$d_x^2 z = 0, \text{ or } \infty, \qquad d_x^2 y = 0, \text{ or } \infty;$$

which are the equations for determining the abscissæ; and which we observe make the expression for the absolute radius of curvature infinite or evanescent, as ought to be the case.

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206. The tangent lines of the given curve $mm'\ldots$ (fig. 59.) will generate a developable surface, whose edge of regression is the curve $mm'\ldots$; and since the tangent line is the intersection of two consecutive osculating planes, this surface would also be generated by the perpetual intersection of the osculating planes. But the lines of intersection of the normal planes are perpendicular to the osculating planes; therefore the angle between two consecutive lines of intersection, that is, two consecutive tangents of $op\ldots$, is equal to the angle between two consecutive osculating planes of $mm'\ldots$ Again, the angle between two consecutive tangents of $mm'\ldots$ is equal to the angle between its corresponding normal planes, that is, to the angle between two consecutive osculating planes of $op\ldots$ If therefore we call the angle between two consecutive tangents of a curve of double curvature its *first flexion*, and that between two consecutive osculating planes its *second flexion*, we may enunciate the above properties as follows. The first flexion of a curve of double curvature is equal to the second flexion of the edge of regression of the developable surface generated by its normal planes; and the first flexion of the latter, is equal to the second flexion of the former.

207. Every curve, whether plane or of double curvature, has an infinite number of evolutes, all of which lie on the developable surface generated by the perpetual intersection of the normal planes.

Recurring to fig. 59, since kh is perpendicular to the plane through two consecutive sides mm', $m'm''$, of the polygon, any point f in it is equidistant from the middle points of the sides g and g'. If therefore we draw gf in the first normal plane, and $g'ff'$ in the second, the lines gf, $g'f$ are inclined at the same angle to kh; and if with center f and radius fg a circle be described, it will touch the two consecutive sides mm', $m'm''$ in their middle points g and g', and when the describing radius comes to g' it will be confounded with $g'f'$. In like manner if we draw $g''f'f''$ in the third normal plane, the lines $g'f'$, $g''f'$ are equally inclined to $k'h'$; and a circle described from f' with radius $f'g'$ will touch the consecutive sides $m'm''$, $m''m'''$ in

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their middle points, and the describing radius will be confounded with $g''f''$ at the point g''. By continuing this construction the polygon $ff'f''\ldots$ will be formed, by unwinding a thread gf from which, circular arcs, touching every two consecutive sides in their middle points, will be described. Therefore when the number of sides is indefinitely increased, the polygon $ff'f''\ldots$ will become a curve traced on the developable surface formed by the intersections of the normal planes; and by unwinding a string from it, the proposed curve of double curvature will be traced out. Also since the direction of the initial radius gf is arbitrary, any other direction would have produced a curve endowed with the same properties as $ff'f''\ldots$; it follows therefore that every curve has an infinite number of evolutes all contained on the surface generated by the perpetual intersections of its normal planes; in the case of a plane curve all the evolutes, except that in its own plane, are curves of double curvature, traced on the right cylinder whose base is the evolute in its own plane.

208. Since $\angle g'f'k' = g''f'k' = h'f'f''$, if the plane hk' were brought into the same plane with $h'k''$ by being turned about $k'h'$, ff' would be brought into the same straight line with $f'f''$; and since the same may be proved successively of all the other sides of the polygon $ff'f''\ldots$, it follows that if the surface formed by the intersections of the normal planes were developed, the evolute $ff'f''\ldots$ would become a straight line, and therefore would be formed by stretching a thread in the direction gf and applying it freely to that surface. Hence if from any point in a curve of double curvature, any line be drawn touching the surface formed by the normal planes, and be applied freely to that surface, it will form an evolute of the proposed curve; and will be the shortest line that can be drawn between any two points of the surface, through which it passes.

209. It is to be observed that the absolute centers of curvature are not situated on an evolute of the curve. For if $g'q'$ be perpendicular to $k'h'$, then it cannot pass through the point q; because it would then coincide with $g'q$, and

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therefore cut kh at right angles, which would require that kh and $k'h'$ should be parallel; and this can never happen as long as the proposed curve is of double curvature. Hence the consecutive absolute radii of curvature gq, $g'q'$, since they are in different planes, and do not meet one another in the common intersection of those planes, therefore they do not meet one another at all; and therefore cannot be consecutive tangents of the same curve; that is, the locus of the absolute center of curvature is not an evolute.

210. To find the equations to any proposed evolute of a curve of double curvature.

As we know the equation to the surface on which all the evolutes lie (Art. 201.), it only remains to find the equation which particularizes a given evolute. Let $x_,$, $y_,$ be the co-ordinates of a point in the projection of an evolute, x, y, the co-ordinates of the corresponding point in the projection of the curve; then a tangent to the projection of the evolute, must pass through the corresponding point in the projection of the curve;

$$\therefore \; d_{x_,} y_, = \frac{y_, - y}{x_, - x}.$$

This joined to the two equations

$$x_, - x + (y_, - y)\, y' + (z_, - z)\, z' = 0,$$

$$(y_, - y)\, y'' + (z_, - z)\, z'' - (1 + y'^2 + z'^2) = 0,$$

which belong to the generating line of the developable surface containing the evolutes, will give by eliminating x the two equations to the evolute, one of them being a differential of the first order; and its integral will introduce an arbitrary constant, by means of which the evolute may be made to satisfy that condition which particularizes it; as, for instance, to pass through a given point of the developable surface on which it is traced.

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211. I. As a first example of this theory, take the curve of double curvature, resulting from the intersection of a sphere and cylinder, considered in Prob. 1. Art. 132.

Its equations are

$$y^2 = 2ax - x^2, \quad z^2 = 4a^2 - 2ax,$$

$$\therefore\ y' = \frac{a-x}{y}, \quad y'' = -\frac{a^2}{y^3}, \quad z' = -\frac{a}{z}, \quad z'' = -\frac{a^2}{z^3}.$$

Hence the equation to the normal plane, as we have seen, is

$$x_1 + y_1 \frac{a-x}{y} = z_1 \frac{a}{z} \ldots\ldots (1).$$

To obtain the equations to the line in which this is intersected by the consecutive normal plane, we must join to it, the derived equation with respect to x, namely,

$$y_1 \left(-\frac{1}{y} - \frac{a-x}{y^2}\, d_x y\right) = -z_1 \frac{a}{z^2}\, d_x z;$$

or, by substitution,

$$\frac{y_1}{y^3} + \frac{z_1}{z^3} = 0, \ldots\ldots (2);$$

and to find the equation to the surface generated by the perpetual intersection of the normal planes, we must eliminate x, y and z, between these two equations, and the equations to the curve.

Now from equation (2),

$$\left(-\frac{y_1}{z_1}\right)^{\frac{2}{3}} = \frac{y^2}{z^2} = \frac{x}{2a}, \quad \text{or } \frac{x}{a} = 2\left(\frac{y_1}{z_1}\right),$$

and equation (1) may be written

$$\frac{x_1}{z_1}\frac{y}{a} + \frac{y_1}{z_1}\frac{a-x}{a} - \frac{y}{z} = 0,$$

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$$\text{or } \frac{x_{,}}{z_{,}}\sqrt{\frac{x}{a}\left(2-\frac{x}{a}\right)}+\frac{y_{,}}{z_{,}}\left(1-\frac{x}{a}\right)+\left(\frac{y_{,}}{z_{,}}\right)^{\frac{1}{3}}=0.$$

Hence substituting for $\frac{x}{a}$ its value, and reducing, we find for the equation to the surface

$$2\left(x_{,}-y_{,}^{\frac{1}{3}}\sqrt{z_{,}^{\frac{2}{3}}-y_{,}^{\frac{2}{3}}}\right)=z_{,}^{\frac{1}{3}}\sqrt{z_{,}^{\frac{2}{3}}-y_{,}^{\frac{2}{3}}}.$$

Again, to find the expression for the radius of curvature,

$$1+y'^2+z'^2=\frac{a}{2y^2}(2a+x)$$

$$(y'')^2+(z'')^2+(y''z'-z''y')^2=\left(\frac{a}{2y^2}\right)^3(10a+3x);$$

$$\therefore \text{ radius of curvature}=\frac{(2a+x)^{\frac{3}{2}}}{\sqrt{10a+3x}}.$$

II. On a given surface, to trace a curve of such a nature, that its involute shall be a plane curve.

The curve of constant inclination, that is, one whose tangent line is inclined at a constant angle to the plane of xy, will have this property. For if s denote the length of the arc, intercepted between the plane of xy and a point whose ordinate is z, and γ = the constant angle which the tangent line makes with the axis of z, then $d_{,}z=\cos\gamma$;

$$\therefore z=s\cos\gamma, \text{ and } s=z\sec\gamma$$

= the length of the tangent line between the point of contact and plane of xy; hence if a string be applied to the curve of equal inclination, and then be unwound from it, beginning from the plane of xy, and be kept stretched in the direction of a tangent, its extremity will be always in the plane of xy, and will describe a curve, which is manifestly the involute of the projection of the curve of equal inclination on the same plane; for the projection of the tangent line will touch the projection of the curve, and will be of the same length. We found the equations to the curve of equal inclination at page 128.

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Ex. Let the surface be a right cone, and the origin in its vertex;

$$\therefore\ \rho = z \tan \alpha, \quad \therefore\ (\cot \alpha)^2 = (\cot \gamma)^2 \cdot \frac{\rho^2}{\rho^2 - p^2},$$

$$\text{or}\ \ p = \rho \sqrt{1 - (\tan \alpha \cot \gamma)^2},$$

the equation to an equiangular spiral. Hence it follows that this curve cuts the generating line of the cone under a constant angle, that is, it is the conical helix. Hence if a thread be applied to the surface of a cone, according to a helix traced on it, and then be unwound beginning from the vertex, its extremity will always be found in a plane perpendicular to the axis through the vertex, and will trace out an equiangular spiral on that plane, which is also the involute of the projection of the conical helix on the same plane, and therefore similar to it.

III. If a uniform and flexible string be suspended from two points in the surface of a vertical cylinder whose base is a circle, it will form itself into a curve of double curvature, such, that the involute lies on the surface of a sphere.

Let A (fig. 61.), be the lowest point of the catenary, AB the radius of curvature at that point; take the horizontal plane drawn through B for the plane of xy, and the axis of the cylinder for that of z, and let CM, MN, NP be the coordinates of any point P in the curve.

Draw NT perpendicular to the tangent PT and join CT; then PT is perpendicular to the plane CTN, for a line drawn through N parallel to PT, and therefore coinciding with the tangent plane of the cylinder would be at right angles both to TN, CN, and therefore to the plane passing through them; and CNT is a right angle, therefore

$$CT^2 = CN^2 + NT^2 = CB^2 + BA^2 = CA^2,$$

because the perpendicular upon the tangent from the foot

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of the ordinate, is constant by the nature of the catenary. Hence the locus of T is on a sphere radius CA, and since CTP is a right angle, therefore PT touches the sphere. But PT = arc AP; if therefore a string be unwound from AP beginning from A, and be always kept stretched in the direction of a tangent, its extremity will trace out a curve situated on a sphere and the string itself will be a tangent to the sphere.

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SECTION IX.

ON THE CURVATURE OF SURFACES.

212. To find the requisite conditions for a contact of the first, second, &c. order, between two surfaces.

If two surfaces, referred to the same origin and axes, pass through the same point, the co-ordinates of which are x, y, z; and if we change x into $x+h$, and y into $y+k$, the equation to the first surface will give for the value of the new ordinate,

$$z + ph + qk + \tfrac{1}{2}(rh^2 + 2shk + tk^2) + \&c.$$

and the equation to the second surface

$$z + Ph + Qk + \tfrac{1}{2}(Rh^2 + 2Shk + Tk^2) + \&c.;$$

the distance of the surfaces, measured in the direction of their ordinates, will therefore be expressed by

$$(P-p)h + (Q-q)k + \tfrac{1}{2}\{(R-r)h^2 + 2(S-s)hk + (T-t)k^2\} + \&c.$$

If we suppose the equation to the second surface to contain a certain number of arbitrary constants, we may determine them so as to make the first terms of this distance vanish; and it will follow that any other surface, for which these terms do not disappear, cannot be situated between the two former, with reference to the points which are contiguous to their common point; at least so long as we take h and k so small, that the sum of the terms of the first order may be more considerable than that of all the terms of succeeding orders. When we have $P-p=0$, $Q-q=0$, the surfaces will have a contact of the first order; if besides these, we have $R-r=0$, $S-s=0$, $T-t=0$, the contact will be of the second order, and so on.

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213. Let therefore $V = 0$ be an equation between three variables x', y', z' and a certain number of arbitrary constants, and according to the usual notation, let P, Q, R, &c. denote the partial differential coefficients of z' with respect to x' and y'; by the determination of the constants we may make this surface have with one completely given (the co-ordinates of which we shall represent by x, y, z, and the partial differential coefficients of z, by p, q, &c.) a contact of an order which will depend on their number. The first condition to be satisfied is that the surfaces may have a common point, that is, that by changing x' into x, and y' into y, in $V = 0$, we may find $z' = z$. Besides this, for a contact of the first order, upon making the same substitutions for x' and y', we must have $P = p$, $Q = q$; and for one of the second, besides all the foregoing conditions, we must have $R = r$, $S = s$, $T = t$.

Hence it appears that a contact of the first order requires three arbitrary constants, and one of the second, six; and in order to have a contact of the n^{th} order, since the terms of three, four, &c. dimensions in h and k in the development of the difference of the ordinates in the preceding Article are, in number, 4, 5, &c., the number of disposable constants must be

$$1 + 2 + 3 + \&c. + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

214. But if we confine our attention to the sections of the surfaces made by a plane containing their common ordinate z, and inclined to the plane of zx at an angle whose tangent is m, then, making $k = mh$, the difference of the consecutive ordinates in these sections will be

$$\{P - p + (Q-q)m\}h + \{R - r + 2(S-s)m + (T-t)m^2\}\tfrac{1}{2}h^2 + \&c.;$$

hence if the surfaces have a common point and a common tangent plane at that point, that is, if upon making $x' = x$, and $y' = y$, we have $z' = z$, $P = p$, $Q = q$; then the first term of the above difference will disappear, and the second will also vanish if

$$-r + 2(S-s)m + (T-t)m^2 = 0.$$

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Hence four disposable constants in the equation to a surface are sufficient to make it have with one completely given, a contact of the first order at a proposed point; and also to establish a contact of the second order between the sections of the surfaces made by a given plane containing their common ordinate.

215. Previously to applying these considerations to estimate the curvature of any surface at a proposed point, we may observe that we cannot attempt to assimilate the curvature of a surface in general to that of a sphere, because in the latter the curvature is uniform about the same normal, whereas for surfaces in general, that is far from being the case. The mode of proceeding must therefore be to imagine several planes drawn through the normal at the point under consideration, to calculate the radius of curvature of each of these sections at that point, and by comparing them to judge of the greater or smaller curvature of the surface in those directions about the point, as well as of the sense in which the curvature is turned; for, as we know, certain surfaces contiguous to any point, are situated partly above and partly below the tangent plane at that point. In thus making different planes pass through the same point of a surface, so that some contain the normal to the surface at that point, and some do not, we shall find the radii of curvature relative to that point, both of the normal and oblique sections, bearing remarkable relations to one another, independent of the particular form of the surface. We shall begin by demonstrating the simple relation which exists between the radii of curvature of a normal and an oblique section, made by planes passing through the same tangent line to a surface, first noticed by Meunier. We shall then, by the doctrine of contacts, deduce the radius of curvature of any normal section of a surface; and afterwards prove Euler's theorems relative to the curvatures of the principal normal sections of a surface.

216. If a normal and an oblique section of a surface be made by planes passing through the same tangent line

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to the surface, the radius of curvature of the oblique section is equal to the projection on its plane, of the radius of curvature of the normal section.

Let the tangent plane to the surface be the plane of xy, the point of contact the origin, and the tangent line the axis of x; then the normal to the surface will be the axis of z; let OP (fig. 62.) be the normal section in the plane of zx, OP' the oblique section made by a plane $z'Ox$ through Ox, and inclined to the normal section at an $\angle z'Oz = \theta$; NP, NP', ordinates to the two curves corresponding to the common abscissa $ON = h$; also let $NP = z$, and let h, k, z' be co-ordinates of P'. Then, (assuming that in any plane curve when the axis of the abscissæ is a normal at the origin, the radius of curvature at the origin is equal to $\frac{1}{2}$ limit of $\left(\frac{(\text{ordin.})^2}{\text{abscissa}}\right)$ we have

$$R = \text{radius of curvature of } OP \text{ at } O = \tfrac{1}{2} \text{ limit of } \frac{ON^2}{NP},$$

$$R' = \text{radius of curvature of } OP' \text{ at } O = \tfrac{1}{2} \text{ limit of } \frac{ON^2}{NP'},$$

$$\therefore \frac{R}{R'} = \text{limit of } \frac{NP'}{NP} = \text{limit of } \frac{z' \sec\theta}{z} = \sec\theta \,.\, \text{limit of } \frac{z'}{z}.$$

But since the plane of xy is the tangent plane at O,

$$p = 0, \quad q = 0, \quad \text{and} \quad z' = \tfrac{1}{2}rh^2 + shk + \tfrac{1}{2}tk^2 + \&c.$$

and making $k = 0$,

$$z = \tfrac{1}{2}rh^2 + \tfrac{1}{6}d_x r \,.\, h^3 + \&c.;$$

$$\therefore \frac{R}{R'} = \sec\theta \,.\, \text{limit of } \frac{r + 2s\frac{k}{h} + t\left(\frac{k}{h}\right)^2 + \&c.}{r + \frac{1}{3}d_x r \,.\, h + \&c.} = \sec\theta,$$

because limit of $\frac{k}{h} = 0$, since Ox is a tangent to the projection of OP' on the plane of xy;

$$\therefore R' = R\cos\theta.$$

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Cor. Hence it follows that the osculating circles of all sections of a curve surface that have a common tangent line, are situated on the surface of a sphere the radius of which is the radius of curvature of the normal section passing through the same tangent; for if OA be the diameter of the circle of curvature of the normal, and OA' that of the oblique section, it is manifest that on account of the right angle $AA'O$, the latter circle is situated on a sphere whose diameter is AO.

217. To find the radius of curvature of any normal section of a surface at a given point, in terms of the co-ordinates of that point.

Let x, y, z be the co-ordinates of the given point P (fig. 63) of the surface, p, q, r &c. the partial differential coefficients of z expressed in terms of those co-ordinates; PT the tangent line through which the normal section is to pass, which, since it lies in the tangent plane at P, will be determined by the equation to its projection on the plane of xy, viz. $y'-y=m(x'-x)$. Then if we determine a sphere having a contact of the first order with the surface at P, and whose section by the vertical plane PQT has a contact of the second order with the section of the surface by the same plane, all planes passing through PT will cut the sphere in circles, which are the circles of curvature to the corresponding sections of the surface, and therefore the radius of the sphere will be the radius of curvature of the normal section; and this we are enabled to do, because the four disposable constants in the equation to a sphere will enable us to satisfy the four requisite conditions, which are, that upon making $x'=x$ and $y'=y$ in the equation to the sphere, we must have

$$z'=z, \quad P=p, \quad Q=q, \quad R-r+2(S-s)m+(T-t)m^2=0.$$

Now the equation to the sphere gives, by differentiation,

$$(x'-a)^2+(y'-\beta)^2+(z'-\gamma)^2=R^2,$$

$$x'-a+P(z'-\gamma)=0, \quad y'-\beta+Q(z'-\gamma)=0,$$

$$1+P^2+R(z'-\gamma)=0, \quad PQ+S(z'-\gamma)=0, \quad 1+Q^2+T(z'-\gamma)=0;$$

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hence, there results between the constants a, β, γ, R, and the co-ordinates x, y, z, the following relations

$$(x-a)^2+(y-\beta)^2+(z-\gamma)^2=R^2,$$

$$x-a+p(z-\gamma)=0, \quad y-\beta+q(z-\gamma)=0,$$

$$\frac{1+p^2}{z-\gamma}+r+2\left(\frac{pq}{z-\gamma}+s\right)m+\left(\frac{1+q^2}{z-\gamma}+t\right)m^2=0.$$

The latter gives

$$z-\gamma=-\frac{1+p^2+2pqm+(1+q^2)m^2}{r+2sm+tm^2}\ldots\ldots\ldots(1);$$

and then a, β, R are known from the equations

$$x-a=-p(z-\gamma), \quad y-\beta=-q(z-\gamma), \quad R^2=(z-\gamma)^2(1+p^2+q^2);$$

$$\therefore R=\frac{\sqrt{1+p^2+q^2}\{1+p^2+2pqm+(1+q^2)m^2\}}{r+2sm+tm^2}.$$

Hence we have determined the radius and co-ordinates of the center of the circle of curvature of the normal section at a point xyz, whose intersection with the tangent plane at that point is projected into a line represented by the equation

$$y'-y=m(x'-x).$$

Cor. Suppose the tangent line to make angles λ, μ, ν with the axes of x, y, z, and let s be the length of the arc of the normal section, then (Art. 28 and 131),

$$m=\frac{\cos\mu}{\cos\lambda}, \quad \text{and}$$

$$(d_s z)^2=1+(d_s y)^2+(d_s z)^2=1+m^2+(p+mq)^2=\left(\frac{1}{\cos\lambda}\right)^2;$$

hence, by substitution, we have for the radius of curvature

$$R=\frac{\sqrt{1+p^2+q^2}}{r(\cos\lambda)^2+2s\cos\lambda\cos\mu+t(\cos\mu)^2}.$$

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218. Hence we can find the conditions, in order that the curvatures of the normal sections of a surface at a proposed point may be all in the same direction, or in opposite directions; that is, in order that the surface may be convex or non-convex about a proposed point.

If in the value of R we always take the radical positive, as the numerator $1+m^2+(p+mq)^2$ is incapable of changing its sign, the sign of R will depend upon that of its denominator $r+2sm+tm^2=\frac{1}{t}\{(mt+s)^2+rt-s^2\}$; if therefore x, y, z the co-ordinates of the proposed point be such that $rt-s^2>0$, then R cannot change its sign for any value of m, and the surface contiguous to that point will be situated entirely on the same side of the tangent plane, or will be convex at that point; and if this be true for every point, as in the ellipsoid, the surface will be entirely convex. If, on the contrary, $rt-s^2<0$, there are two values of m for which the denominator will vanish, and R will be infinite and will change its sign, as m passes through each of these values; therefore the surface will have opposite curvatures about the point in question, or will be non-convex. Thus in the surfaces considered (Art. 176), of which the equation is

$$q^2r-2pqs+p^2t=0, \quad \text{or} \quad (qr-ps)^2+p^2(rt-s^2)=0,$$

$rt-s^2$ is necessarily negative, and therefore surfaces of this class are non-convex at every point. For developable surfaces, where $rt-s^2=0$, the denominator of R is a perfect square, and consequently retains the same sign; therefore the radius of curvature will have the same sign for every point of the surface; only it will at every point be infinite when $mt+s=0$, the direction so determined evidently coinciding with that of the generating line passing through the point.

219. For the same point of the surface, the expression for R varies with m, and we may find what value of m will make it a maximum or minimum by putting the differential coefficient of R with respect to m (for the

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quantities p, q, r &c. are in general independent of the position of the tangent line which fixes the normal section) or rather of $z - \gamma$ on which it depends, equal to 0; hence putting equation (1) (Art. 217), under the form

$$(z-\gamma)\{r+2sm+tm^2\}+1+p^2+2pqm+(1+q^2)m^2=0,$$

we have $$(z-\gamma)(s+tm)+pq+(1+q^2)m=0;$$

therefore eliminating successively $z-\gamma$ and m from these equations, we have

$$\{(1+q^2)s-pqt\}m^2+\{(1+q^2)r-(1+p^2)t\}m-\{(1+p^2)s-pqr\}=0,$$

$$(rt-s^2)(z-\gamma)^2+\{(1+p^2)t-2pqs+(1+q^2)r\}(z-\gamma)+1+p^2+q^2=0.$$

These equations being of the second degree, it follows that in general the radius of curvature of the normal section, as the cutting plane revolves about the normal, will have only one maximum and one minimum value, which, in absolute magnitude, may be two maxima or two minima, when the curvature of the surface changes its sign about the given point. These two normal sections, one corresponding to the least, and the other to the greatest radius of curvature, are called the *principal sections* of the surface relative to the point at which the normal is drawn, and the corresponding radii the principal radii of curvature.

Cor. Hence the former of the above equations enables us, for each point of the surface, to determine the directions of the principal sections, and we might also shew from it, that these directions are at right angles to one another. To determine however the angle between the principal sections in a simple manner, let us suppose the plane of xy to be coincident with the tangent plane at P, or only parallel to it, (this of course we may do without at all altering the form of the surface or the positions of the tangents to the principal sections) then we must put $p=0$, $q=0$, and the equation becomes

$$m^2+\frac{r-t}{s}m-1=0,$$

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the two roots of which are always possible and satisfy the condition

$$m'm'' = -1;$$

therefore the principal sections always exist and are at right angles to one another. Also the same supposition gives for the radius of curvature of any normal section (Art. 217),

$$R = \frac{1 + m^2}{r + 2sm + tm^2}.$$

Having however made the foregoing process answer its purpose of giving the radius of curvature of any normal section of a surface in terms of the co-ordinates of the point, we shall not deduce from it all the important results which it is capable of furnishing, but obtain them by a direct and much simpler method.

220. The sum of the curvatures of any two normal sections at right angles to one another, is a constant quantity at the same point of a surface.

Let AO, the normal at any point A of a curve surface, be the axis of z, and xAy, the tangent plane at the same point, the plane of xy (fig. 64). Let AB be a section of the surface made by any plane passing through AO, AC the corresponding section of the tangent plane, which therefore touches the plane curve AB at A; and let $y = \tan\theta . x$ be the equation to AC, so that $\theta = \angle NAC$. Then if $AO = R$ be the radius of curvature of the plane curve AB at A, and BC be parallel to AO, we have by a well-known theorem

$$R = \tfrac{1}{2} \text{ limit of } \frac{AC^2}{BC}.$$

Let $AN = h$, $NC = k$, be the co-ordinates of C;

$$\therefore BC = \tfrac{1}{2}rh^2 + shk + \tfrac{1}{2}tk^2 + \&c.,$$

for $p = 0$, $q = 0$, since the plane of xy is the tangent plane; also $k = \tan\theta . h$;

$$\therefore R = \tfrac{1}{2} \text{ limit of } \frac{h^2 + k^2}{\tfrac{1}{2}rh^2 + shk + \tfrac{1}{2}tk^2 + \&c.}$$

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$$= \text{limit of} \frac{1 + (\tan\theta)^2}{r + 2s\tan\theta + t(\tan\theta)^2 + \frac{1}{3}d_x r \,.\, h(\tan\theta)^3 + \&c.}$$

$$\text{or } R = \frac{1 + (\tan\theta)^2}{r + 2s\tan\theta + t(\tan\theta)^2};$$

where r, s, t are the values of $d_x^2 z$, $d_x d_y z$, $d_y^2 z$, at the point A derived from the equation to the surface. Hence, if R' be the radius of curvature of another section inclined at an angle $= 90^0 + \theta$ to AN, we have

$$\frac{1}{R} = r(\cos\theta)^2 + 2s\cos\theta\sin\theta + t(\sin\theta)^2,$$

$$\frac{1}{R'} = r(\sin\theta)^2 - 2s\sin\theta\cos\theta + t(\cos\theta)^2;$$

$$\therefore \frac{1}{R} + \frac{1}{R'} = r + t,$$

which (taking, as usual, the inverse of the radius of curvature to measure the curvature) expresses that the sum of the curvatures of any two normal sections at right angles to one another, is a constant quantity at the same point of the surface.

221. Of all sections of a curve surface made by planes drawn through the normal at any point, to determine those of greatest and least curvature at that point; and to shew that they are at right angles to one another.

We have

$$\frac{1}{R} = r(\cos\theta)^2 + 2s\sin\theta\cos\theta + t(\sin\theta)^2,$$

$$\text{or } \frac{2}{R} = r(1 + \cos 2\theta) + 2s\sin 2\theta + t(1 - \cos 2\theta)$$

$$= r + t + (r - t)\cos 2\theta + 2s\sin 2\theta,$$

which is to be a maximum or minimum by the variation of θ; therefore putting its differential coefficient with respect to θ equal to zero, we have

$$-(r - t)\sin 2\theta + 2s\cos 2\theta = 0;$$

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$$\therefore \tan 2\theta = \frac{2s}{r-t} = \tan(180^0 + 2\theta); \quad (1),$$

$$\sin 2\theta = \pm \frac{2s}{\sqrt{(r-t)^2+4s^2}}, \quad \cos 2\theta = \pm \frac{r-t}{\sqrt{(r-t)^2+4s^2}},$$

$$\therefore \frac{2}{R} = r + t \pm \frac{(r-t)^2}{\sqrt{(r-t)^2+4s^2}} \pm \frac{4s^2}{\sqrt{(r-t)^2+4s^2}},$$

$$\text{or} \quad \frac{1}{R} = \tfrac{1}{2}(r+t) \pm \tfrac{1}{2}\sqrt{(r-t)^2+4s^2},$$

which are the values of the reciprocals of the greatest and least radius of curvature; and the positions of the sections to which they belong are fixed by the angles θ and $90^0 + \theta$ obtained from equation (1); consequently the sections of greatest and least curvature at any point of a surface, called the principal sections at that point, are at right angles to one another.

222. The curvature of any normal section is equal to the sum of the curvatures of the principal sections, multiplied respectively by the squares of the cosines of the angles which its plane forms with their planes.

Suppose the axes of x and y to be drawn in the principal planes, then since $\theta = 0$ and $\tan 2\theta = \frac{2s}{r-t}$, we must have $s = 0$; hence

$$\frac{1}{R} = r(\cos\theta)^2 + t(\sin\theta)^2;$$

consequently, if we denote by ρ and ρ' the radii of curvature of the principal sections corresponding to $\theta = 0$ and $\theta = 90^0$, we have

$$\frac{1}{\rho} = r, \quad \frac{1}{\rho'} = t;$$

$\therefore$ substituting, we have for any normal section,

$$\frac{1}{R} = \frac{1}{\rho}(\cos\theta)^2 + \frac{1}{\rho'}(\sin\theta)^2.$$

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This theorem, due to Euler, joined to that of Meunier, contains the whole theory of the curvature of surfaces; for it is hence sufficient to know at any point of a surface the directions and curvatures of the principal sections, to deduce the curvature at that point of every other section, normal or oblique.

223. When the two principal radii have the same sign, the formula

$$\frac{1}{R} = \frac{1}{\rho}(\cos\theta)^2 + \frac{1}{\rho'}(\sin\theta)^2$$

proves that every normal section will have a radius of that sign, and therefore the surface will be entirely on the same side of the tangent plane at the proposed point; also the principal radii will be the maximum and minimum values of the radius of curvature at that point, and the form of the surface will be that represented in fig. 65, where AO, AO' are the principal radii, and AQ is the radius of the intermediate section AP.

When the principal radii are equal, as well as of the same sign, the formula gives $R = \rho$, and therefore all the normal sections have the same curvature, and may be regarded as principal sections; of this we have examples at the vertex of a paraboloid of revolution, and the poles of a spheroid.

224. When the principal radii are of contrary signs, ρ positive and ρ' negative for instance,

$$\frac{1}{R} = \frac{1}{\rho}(\cos\theta)^2 - \frac{1}{\rho'}(\sin\theta)^2,$$

which vanishes, or R is infinite, when θ has such a value ω that $\tan\omega = \pm\sqrt{\frac{\rho'}{\rho}}$; hence for values of θ between 0 and $\pm\omega$ R is positive, or the section situated above the tangent plane; and for all other values, R is negative, or the section situated below the tangent plane; also ρ will be the minimum of the positive radii, and ρ', numerically, the

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minimum of the negative radii. Hence at the proposed point, the form of the surface will be that represented in fig. 66, DBE being the intersection of the surface with a sphere center A, AE, AF sections having respectively a positive and negative radius, and AB the section made by the limiting normal plane through AH. Thus in the hyperboloid of one sheet, we know that the tangent plane at any point cuts the surface in two straight lines, limiting the normal planes which give positive and negative radii of curvature, or which separate the convex and concave parts of the surface. In surfaces of a higher order, the limiting normal planes will cut them in curves, having with the corresponding sections of the tangent planes a contact at least of the second order, since the curvatures of those sections are infinite. In developable surfaces, if we take a principal plane at any point for that of zx, then $s = 0$, and since $rt - s^2 = 0$, $r = 0$ or $t = 0$; therefore one of the principal radii is infinite. In fact, we know that the tangent plane to a developable surface does not cut it, but touches it along a generating line; and that this generating line is a principal section whose radius of curvature is a maximum and infinite, whilst the minimum radius belongs to the section made perpendicular to the generating line. Hence, the curvature of any surface at every one of its points may be assimilated to that of a surface of the second order at one of its vertices, as will be seen in the following Article, where we shall take a paraboloid, although an ellipsoid would do equally well.

225. To determine a paraboloid of the second order, which shall have at its vertex, a complete contact of the second order with a given surface at a proposed point.

Let A (figs. 65 and 66,) be the given point of the surface, Az the normal, and let the principal sections meet the tangent plane in Ax, Ay, which lines take for the axes. Also, let

$$z = \frac{x^2}{2\rho} \pm \frac{y^2}{2\rho'}$$

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be the equation to a paraboloid, ρ, ρ' being the principal radii of curvature of the proposed surface at A. Then if AP be a section of this paraboloid through its axis inclined at an $\angle\ \theta$ to zx, and $AN = r$, $PN = z$, be co-ordinates of P, the equation to AP, putting $r\cos\theta$ for x and $r\sin\theta$ for y, (Art. 101) is

$$z = \tfrac{1}{2} r^2 \left\{ \frac{1}{\rho} (\cos\theta)^2 \pm \frac{1}{\rho'} (\sin\theta)^2 \right\},$$

which represents a parabola, the reciprocal of whose semi latus rectum, and therefore of the radius of curvature at its vertex,

$$= \frac{1}{\rho} (\cos\theta)^2 \pm \frac{1}{\rho'} (\sin\theta)^2.$$

Hence the curvature of every section of the paraboloid is the same as that of the corresponding section of the surface, and therefore the paraboloid has a complete contact of the second order with the surface.

Cor. What we here assume, viz. that if two surfaces, having a common point and common normal at that point, have the curvatures of all normal sections equal, (or, which comes to the same thing, have their principal sections coincident and equally curved,) they have a complete contact of the second order at that point, agrees with Art. (213); for this, from the general expression for the radius of curvature of any normal section, requires that z and its differential coefficients p, q, r, s, t, should have equal values in the equations to the two surfaces, when in them we substitute the co-ordinates x and y of the point under consideration.

226. In the equation to the paraboloid, suppose z constant, and $= c$, therefore

$$\frac{x^2}{2\rho c} \pm \frac{y^2}{2\rho' c} = 1;$$

this is the equation to a section of the paraboloid, or to a section of the surface if c be indefinitely small. Hence it appears that if the surface be cut by a plane parallel to

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the tangent plane at any point, and indefinitely near to it, the section is ultimately a curve of the second order whose centre is in the normal, and axes in the planes of greatest and least curvature; also the square of the diameter in which the curve is intersected by any plane drawn through the normal, is proportional to the radius of curvature of the corresponding section of the surface.

This curve has been called by Dupin, the *Indicatrix* of the surface, because it indicates the directions of the curvatures; if at any point it be an ellipse, ρ and ρ' must have the same sign, that is, the curvatures are in the same direction; if a hyperbola, the curvatures are in opposite directions; and if a circle, $\rho = \rho'$, and the curvature of every normal section is the same. For instance, it has been shewn that an ellipsoid may be generated in two ways, by a circle of variable radius moving parallel to itself; consequently there are four points on its surface at which a plane, parallel and indefinitely near to the tangent plane, cuts it in a circle; that is, the indicatrix at those points is a circle, and therefore the two radii of curvature equal. These points are called *umbilici*, and are symmetrically placed in the four angles of the principal section containing the greatest and least axes. It is manifest that at these points, a sphere can have a complete contact of the second order with the surface.

227. As two lines in space will not intersect unless the constants which enter into their equations satisfy a certain equation of condition, so if from any point in a curve surface at which we have drawn a normal we pass to a contiguous point, the normal at the latter point will not intersect that at the former, however near the points are to one another, unless the second point be taken in such a direction as to satisfy the equation of condition for the intersection of the normals. The consideration of consecutive normals throws great light on the subject of the curvature of surfaces, as will appear from the following propositions.

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228. Having given a point on a curve surface, to find the directions in which we must pass to consecutive points, in order that the corresponding normals may intersect.

Take the normal at the given point for the axis of z, and the tangent plane for that of xy, and let h, k, l be co-ordinates of a contiguous point situated in a section through the normal made by a plane inclined at an angle θ to that of zx, so that $k = h \tan\theta$; then the equations to the normal at that point are

$$x' - h + p_1(z' - l) = 0, \quad y' - k + q_1(z' - l) = 0,$$

where p_1, q_1 denote certain functions of h and k (p and q denoting the values of $d_x z$ and $d_y z$ at the origin, and therefore each = 0). Then in order that this line may meet the axis of z, its two equations must agree in giving the same value for z' when x' and $y' = 0$, that is,

$$z' = \frac{h}{p_1} + l \quad \text{and} \quad z' = \frac{k}{q_1} + l,$$

must be the same. But if the points be consecutive, then the ultimate values of z' must be the same when $h = 0$, that is, (expanding p_1 and q_1),

$$z' = \text{limit of } \frac{h}{rh + sk + \&c.} = \frac{1}{r + s\tan\theta}$$

$$\text{and } z' = \text{limit of } \frac{k}{sh + tk + \&c.} = \frac{\tan\theta}{s + t\tan\theta}$$

(r, s, t being the values of $d_x^2 z$, $d_x d_y z$, $d_y^2 z$ at the origin) must be identical;

$$\therefore\ s + t\tan\theta = r\tan\theta + s(\tan\theta)^2,$$

$$\text{or } \tan 2\theta = \frac{2\tan\theta}{1 - (\tan\theta)^2} = \frac{2s}{r - t},$$

an equation which being the same as that for finding the positions of the principal sections through the origin, shews that there are two, and only two, directions at right angles

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to one another in which we may pass from a proposed point in a curve surface to a consecutive point, so that the normals may intersect; and that these directions coincide with the principal sections through the proposed point. Also the value of

$$\frac{1}{s'} = r + s\tan\theta = \tfrac{1}{2}\left\{r + t \pm \sqrt{(r-t)^2 + 4s^2}\right\}$$

shews (Art. 221.) that the points where the normal is intersected by consecutive normals, are the centers of curvature of the principal sections.

229. This property of consecutive normals enables us in some cases to determine at once the principal sections of a surface at a proposed point; for instance, in surfaces of revolution the plane of the generating curve is necessarily one of those sections, because in it consecutive normals intersect, and therefore a plane through the normal perpendicular to it is the other. The first series of consecutive normals intersect in the evolute of the generating curve, the second in the axis; hence, in a surface of revolution, the loci of the intersections of consecutive normals, or of the centers of curvature of the principal sections, are the axis of revolution and the surface formed by the revolution about that axis of the evolute of the generating curve. Hence we can find immediately the radius of curvature of any section of a surface of revolution, the principal radii being the radius of curvature of the meridian, and the portion of the normal intercepted between the proposed point and the axis of revolution.

230. A series of points on a surface determined by the condition that the normals at any two consecutive ones intersect is called a *line of curvature* of the surface. Every point of a surface, as appears from the above investigation, is situated on two curves of this kind, whose directions coincide with those of the principal sections through that point, and cut one another at right angles. Thus in starting from a point P of a surface (fig. 68.), there are two

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contiguous points P' and Q the normals at which will intersect the normal at P; and if of these we take only that which is in the same direction with KP, namely, P', and then advance in the same direction to the contiguous point whose normal intersects the normal at P', and so on continually, we shall obtain a first line of curvature of the surface $KPP'n$; the second line of curvature which passes through P will be obtained in the same manner, and will be $RPQE$; and as these constructions may be repeated for every point of the surface, we shall thus form two series of lines of curvature dividing the surface into curvilinear quadrilaterals whose sides in space cut one another at right angles. It must be observed that the lines of curvature which pass through any point and the principal sections through the same point, although they have common tangent lines at that point, will not usually coincide. Thus in a surface of revolution the meridian PB (fig. 67.) and the section through the normal perpendicular to the meridian QPG are the principal sections at P; but only one of them is a line of curvature through P, viz. the meridian; the other line of curvature being manifestly the parallel PD, which has a common tangent with PQ at P.

231. To find the differential equation to the lines of curvature which pass through a given point of a curve surface.

The equations to the normal at a point xyz of a curve surface, are

$$x' - x + p(z' - z) = 0, \quad y' - y + q(z' - z) = 0;$$

and at a contiguous point $x + h$, $y + k$, $z + l$, the equations to the normal are

$$x' - x - h + p_1(z' - z - l) = 0, \quad y' - y - k + q_1(z' - z - l) = 0;$$

p_1, q_1, denoting the same functions of $x + h$, $y + k$, that p and q are of x and y, as derived from the given equation to the surface $z = f(x, y)$. If these two normals intersect, their equations must be satisfied by the same

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values of x', y', z', which will be the co-ordinates of their point of intersection; eliminating therefore x', y', z' between the four preceding equations, we shall have an equation of condition expressing that the normals intersect, and which will establish a relation between h and k or fix the direction in which we must pass from the proposed point to a consecutive point in order that the normals may intersect. Subtracting the first equations from the second, we have

$$-h+(p_1-p)(x'-x)-p_1 l=0, \quad -k+(q_1-q)(x'-x)-q_1 l=0,$$

and these equations must agree in giving the same value for x',

$$\therefore \frac{h+p_1 l}{k+q_1 l}=\frac{p_1-p}{q_1-q}.$$

Now let the points be consecutive; then expanding p_1, q_1, and l, and retaining in their developments only those terms which involve the simple powers of h and k, we find

$$\text{limit of } \frac{h+(p+rh+sk)(ph+qk)}{k+(q+sh+tk)(ph+qk)}=\text{limit of } \frac{rh+sk}{sh+tk}.$$

But if θ be the angle which the tangent to the projection of a line of curvature on the plane of xy makes with the axis of x, we have $k=h\tan\theta$ ultimately; hence substituting this in the above equation, and making h vanish, we have

$$\frac{1+p^2+pq\tan\theta}{\tan\theta+pq+q^2\tan\theta}=\frac{r+s\tan\theta}{s+t\tan\theta}.$$

But if we now consider x and y to be the co-ordinates of the projection of the line of curvature, we have $\tan\theta=d_x y$; hence, substituting and reducing,

$$\{(1+q^2)s-pqt\}(d_x y)^2$$
$$+\{(1+q^2)r-(1+p^2)t\}d_x y-\{(1+p^2)s-pqr\}=0.$$

It remains to substitute for p, q, r, s, t their values in terms of x and y derived from the equation to the surface and to integrate; the result will be the equation to the

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projection of the lines of curvature. Since the above equation is of two dimensions in $d_x y$, its integral when completed by the arbitrary constant C, will be of the form

$$C^2 + C\phi(x, y) + \psi(x, y) = 0.$$

Suppose the line of curvature is to pass through a point for which $x = a$, $y = b$; therefore

$$C^2 + C\phi(a, b) + \psi(a, b) = 0,$$

which will give two values of C; and by substituting them successively in the complete integral, we shall have the equations to the two lines of curvature passing through the given point.

232. The points in which the normal at any point is intersected by the consecutive normals, coincide, as we have seen, with the centers of curvature of the principal sections at that point; and the portions of the normal intercepted between these points and the point of the surface at which it is drawn, which, as we have seen, are the radii of curvature of the principal sections of the surface at that point, have been called by Monge *the two radii of curvature of the surface* at that point, as being the radii of two spheres which alone can touch the surface in two consecutive points; for if with the points O, O' (fig. 65.) as centers, and radii OA, $O'A$, we describe spheres, the former will touch the surface along AG and the latter along AD, because consecutive normals to the surface in both these sections meet AO; but if with Q, the center of curvature of any section AP, as center, and radius AQ we describe a sphere, it will not touch the surface along AP, because QP cannot be a normal to the surface, since it intersects AO; only the sphere will have the same tangent plane as the surface at A, and its section made by the plane AQP will have with the corresponding section of the surface a contact of the second order. From the consideration of consecutive normals we can now find the expressions for the lengths of the principal radii of a surface at any point much more easily than by the former method.

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233. To determine the radii of curvature at any point of a surface in terms of the co-ordinates of that point.

The equations to the normal at a point xyz of a curve surface are

$$x' - x + p(z' - z) = 0, \quad y' - y + q(z' - z) = 0,$$

and at a contiguous point $x + h$, $y + k$, $z + l$, the equations to the normal are

$$x' - x - h + p_1(z' - z - l) = 0, \quad y' - y - k + q_1(z' - z - l) = 0.$$

The values of x', y', z' which simultaneously satisfy these four equations are the co-ordinates of the point of intersection of the normals; subtracting the two former from the latter, we have

$$-h + (p_1 - p)(z' - z) - p_1 l = 0, \quad -k + (q_1 - q)(z' - z) - q_1 l = 0,$$

both of which the co-ordinate z' must satisfy. Now let the points be consecutive, and let the second point be situated on a line of curvature the projection of whose tangent on the plane of xy makes an $\angle\theta$ with the axis of x, then $k = h\tan\theta$ ultimately; hence, expanding p_1, q_1, and l in the above equations, substituting for k, and then making $h = 0$, we find the equations

$$(r + s\tan\theta)(z' - z) = 1 + p^2 + pq\tan\theta,$$

$$(s + t\tan\theta)(z' - z) = \tan\theta + pq + q^2\tan\theta,$$

which must be satisfied by the same value of z'; hence, eliminating $\tan\theta$, we must have

$$\frac{1 + p^2 - r(z' - z)}{(z' - z)s - pq} = \frac{(z' - z)s - pq}{1 + q^2 - (z' - z)t},$$

$$\text{or } (rt - s^2)(z' - z)^2 - \{(1 + p^2)t - 2pqs + (1 + q^2)r\}(z' - z)$$

$$+ 1 + p^2 + q^2 = 0 \ldots\ldots (1),$$

the equation which determines $z' - z$, the projection of a principal radius of curvature on the axis of z, so that

$$\rho = -(z' - z)\sqrt{1 + p^2 + q^2}.$$

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Hence if we find the two values of $z' - z$ from the above equation, and substitute them in the value of ρ, we shall obtain the values of the two principal radii of the surface at the proposed point; also knowing $z' - z$, the other co-ordinates x' and y' of the centers of curvature are known from the equations to the normal

$$x' - x + p(z' - z) = 0, \quad y' - y + q(z' - z) = 0.$$

The roots of equation (1) have the same or different signs, according as $rt - s^2$ is positive or negative; therefore the curvatures are in the same or different directions, as we have before seen, according as $rt - s^2$ is positive or negative.

Cor. In order that the two radii of curvature may be equal but of different signs, we must have

$$(1 + p^2)t - 2pqs + (1 + q^2)r = 0.$$

234. In order that the two radii of curvature may be equal but of the same signs, we must have (solving the equation and putting the part under the radical equal to zero)

$$\{(1+p^2)t-2pqs+(1+q^2)r\}^2-4(1+p^2+q^2)(rt-s^2)=0.....(1)$$

$$\text{or } \{(1 + p^2)t - (1 + q^2)r + 2(1 + q^2)r - 2pqs\}^2$$

$$- 4\{(1 + p^2)(1 + q^2) - p^2q^2\}(rt - s^2) = 0;$$

and expanding the first term and reducing, we find

$$\{(1+p^2)t-(1+q^2)r\}^2+4\{(1+p^2)s-pqr\}\{(1+q^2)s-pqt\}=0...(2).$$

$$\text{Let } (1 + p^2)s - pqr = pqu, \quad (1 + q^2)s - pqt = pqv;$$

$$\therefore \{(1 + p^2)v - (1 + q^2)u\}^2 + 4uv = 0;$$

$$\text{but } \left\{(1 + p^2)v - (1 + q^2)u + \frac{2u}{1 + p^2}\right\}^2$$

$$= \{(1 + p^2)v - (1 + q^2)u\}^2 + 4uv - \frac{4(1 + q^2)}{1 + p^2}u^2 + \frac{4u^2}{(1+p^2)^2};$$

$$\therefore \left\{(1+p^2)v-\left(1+q^2-\frac{2}{1+p^2}\right)u\right\}^2 + \frac{(1+p^2)(1+q^2)-1}{(1 + p^2)^2}4u^2=0...(3),$$

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an equation which can only be satisfied by putting

$$u = 0, \qquad v = 0,$$

$$\text{or } (1+p^2)s - pqr = 0, \qquad (1+q^2)s - pqt = 0.$$

From this process it appears generally that the roots of the equations which give the principal radii, and the directions of the principal sections, at any point of a surface are always real; for these roots involve the square roots of the polynomials (1) and (2), which are here shewn to be equivalent to one another and to (3), which is a form that never can become negative.

235. The points of a surface at which the two radii of curvature are the same both in magnitude and sign, are called, as has been stated, umbilici. The co-ordinates of such points must satisfy the double equation

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t};$$

at which we may arrive immediately by observing that, since at such points all normal sections have the same curvature, the expression for the radius of curvature of any normal section,

$$R = \frac{\sqrt{1+p^2+q^2}\,\{1+p^2+2pqm+(1+q^2)m^2\}}{r+2sm+tm^2}$$

must be independent of the quantity m which particularizes the normal sections; and for this it is necessary and sufficient that the coefficients of like powers of m in the numerator and denominator should be equal;

$$\therefore \frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t}.$$

Also at these points the equation which determines the directions of the principal sections at any point, since it may be put under the form

$$vm^2 + \left\{\frac{(1+p^2)v - (1+q^2)u}{pq}\right\}m - u = 0,$$

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becomes identical, and therefore the directions of the principal sections indeterminate; in fact, all the normal sections having the same curvature, each of them may be considered as a principal section.

236. To find whether a given surface admits of umbilici, we must substitute the values of p, q, r, s, t, derived from its equation in the double equation

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t},$$

and examine whether these together with the equation to the surface can be satisfied by real values of x, y, z. Consequently the number of umbilici of à given surface will in general be limited; only if the double equation should be reducible to a single distinct equation, then that equation joined to the equation to the surface will determine a curve on the surface every point of which will be an umbilicus, and which is called a line of spherical curvature, because about each of its points the surface is uniformly curved like a sphere.

237. If through all the points of a line of curvature KPP' (fig. 68.) we draw normals to the surface, since every two consecutive ones intersect, they will form a developable surface whose edge of regression will be the locus of the centers of the first curvature relative to KPP'. Proceeding in a similar manner for each line of the same curvature LQQ', &c., we shall obtain a series of developable surfaces of which the edges of regression will form by their assemblage a surface which is the locus of all the centers of the first curvature, and to which all the normals will be tangents. Also this surface will have a second sheet which will be the locus of the centers of the second curvature, resulting from the edges of regression of the surfaces generated by the normals along the lines of the second curvature RPQ, $MP'Q'$, &c.; and which will be touched by the same normals as the other. To obtain the equation to this surface, we have only to eliminate x, y, z between the equation which

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gives $z' - z$, the projection of a principal radius upon the axis of z (Art. 233.), and the equations to the normal.

When the two sheets intersect, their intersection will be the locus of the centers relative to the line of spherical curvature of the surface under consideration.

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238. We shall terminate this section, by applying the results obtained in it to several particular cases.

I. To find the equation to a plane which shall have a contact of the first order with a surface at a proposed point.

Since the equation to a plane (which here takes the place of $V = 0$, Art. 213) contains only three arbitrary constants, a plane can in general have only a contact of the first order with a surface completely given. We have therefore

$$z' = Ax' + By' + c, \quad P = A, \quad Q = B,$$

from which there results between the constants A, B, c, and the co-ordinates x, y, z of the proposed point in the surface, the relations

$$z = Ax + By + c, \quad A = p, \quad B = q;$$

subtracting therefore the value of z from that of z', and putting for A and B their values, the equation to the plane having a contact of the first order with a given surface at a point xyz, is

$$z' - z = p(x' - x) + q(y' - y).$$

II. The equation to a sphere is

$$(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = \delta^2,$$

and since it contains only four arbitrary constants, a sphere cannot generally have a complete contact of the second

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order with a surface. Suppose it required only to have a contact of the first order;

$$\text{then } d_{x'}z' = -\frac{x'-\alpha}{z'-\gamma}, \quad d_{y'}z' = -\frac{y'-\beta}{z'-\gamma}.$$

Hence there results between the constants α, β, γ, δ, and the co-ordinates x, y, z, of the proposed point of the surface the following relations,

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\delta^2, \quad -\frac{x-\alpha}{z-\gamma}=p, \quad -\frac{y-\beta}{z-\gamma}=q;$$

$$\text{or } x-\alpha+p(z-\gamma)=0, \quad y-\beta+q(z-\gamma)=0,$$

which equations shew that the centers of all the spheres lie in the normal to the point of contact.

$$\text{Also } (z-\gamma)^2\{1+p^2+q^2\}=\delta^2, \quad \text{or } z-\gamma=\frac{\delta}{\sqrt{1+p^2+q^2}};$$

$$\therefore \gamma = z-\frac{\delta}{\sqrt{1+p^2+q^2}}, \quad \alpha = x+\frac{p\delta}{\sqrt{1+p^2+q^2}},$$

$$\beta = y+\frac{q\delta}{\sqrt{1+p^2+q^2}}.$$

Hence the number of spheres, which may have a simple contact with a surface at the same point, is unlimited; the radius of the sphere being assigned, the co-ordinates of its center are given by the above equations.

III. To find the radius of curvature of any normal section, at a given point of an oblate spheroid. (Art. 229.)

Let $l = \angle PGA$, PG being a normal (fig. 67.);

$$\therefore \text{radius of curvature of meridian } PO = \rho = \frac{a(1-e^2)}{\{1-e^2(\sin l)^2\}^{\frac{3}{2}}},$$

$$\text{radius of curv. perp. to meridian } PO' = \rho' = \frac{a}{\sqrt{1-e^2(\sin l)^2}},$$

$$\therefore \rho = \rho' \cdot \frac{1-e^2}{1-e^2(\sin l)^2}.$$

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Let the section PQ make an $\angle(\alpha)$ with the meridian, and let its radius of curvature $= R$,

$$\therefore R = \frac{\rho\rho'}{\rho(\sin\alpha)^2 + \rho'(\cos\alpha)^2}$$

$$= \rho' \frac{1-e^2}{\{1-e^2(\sin l)^2\}(\cos\alpha)^2 + (1-e^2)(\sin\alpha)^2}$$

$$= \frac{a}{\sqrt{1-e^2(\sin l)^2}} \cdot \frac{1-e^2}{1+e^2(\cos l\cos\alpha)^2 - e^2}.$$

IV. To find the umbilici of an ellipsoid.

These are points (Art. 226.) at which, if the ellipsoid were generated by a circle of variable radius moving parallel to itself, the plane of the generating circle would become the tangent plane. Let M (fig. 67.) be one of these points which is necessarily in the principal section containing the greatest and least axes, and let MT be the tangent at M; then (Art. 104.)

$$\tan MTN = \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}} = \frac{c}{a}\frac{x}{\sqrt{a^2-x^2}}$$

if $CN = x$, since the equation to BPM is $z = \frac{c}{a}\sqrt{a^2-x^2}$;

hence the co-ordinates of M, and of the three other umbilici similarly situated in the other quadrants of the principal section, are

$$x = \pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad z = \pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}.$$

V. To determine the lines of curvature of an ellipsoid.

Let the equation be

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1;$$

$$\therefore p = d_x z = -\frac{\gamma}{\alpha}\frac{x}{z}, \quad q = d_y z = -\frac{\gamma}{\beta}\frac{y}{z},$$

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$$r = d_x p = -\frac{\gamma^2}{\alpha\beta x^3}(\beta - y^2), \quad s = d_y p = -\frac{\gamma^2}{\alpha\beta x^3} xy,$$

$$t = d_y q = -\frac{\gamma^2}{\alpha\beta x^3}(\alpha - x^2);$$

therefore, by substitution in the equation of Art. 231,

$$\alpha(\beta - \gamma)\, xy\, (d_x y)^2$$

$$+ \{\beta(\alpha - \gamma)x^2 - \alpha(\beta - \gamma)y^2 - \alpha\beta(\alpha - \beta)\} d_x y - \beta(\alpha - \gamma)\, xy = 0,$$

$$\text{or} \quad A xy\, (d_x y)^2 + (x^2 - Ay^2 - B)\, d_x y - xy = 0;$$

$$\text{calling} \quad \frac{\alpha(\beta - \gamma)}{\beta(\alpha - \gamma)} = A, \quad \frac{\alpha(\alpha - \beta)}{\alpha - \gamma} = B,$$

which will be both positive if we suppose the plane of projection, that is, the plane of xy, to contain the greatest and the mean axis, so that α, β, γ are in order of magnitude.

To integrate this equation, let $d_x y = \frac{xu}{y}$, u being a function of x;

$$\therefore xy\left(A\frac{x^2u^2}{y^2} - 1\right) + \frac{xu}{y}(x^2 - Ay^2 - B) = 0,$$

$$\text{or} \quad Ax^2u^2 - y^2 + u\,(x^2 - Ay^2 - B) = 0,$$

$$\text{or} \quad y^2(1 + Au) = x^2u\,(1 + Au) - Bu;$$

$$\therefore y^2 = x^2u - \frac{Bu}{1 + Au};$$

differentiate, then

$$2y\, d_x y = 2xu = 2xu + x^2\, d_x u - \frac{B}{(1 + Au)^2} d_x u,$$

$$\text{or} \quad d_x u\left(x^2 - \frac{B}{(1 + Au)^2}\right) = 0;$$

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$$\therefore\ u = C,\ \text{or}\ x = \frac{\sqrt{B}}{1 + Au};$$

$$\therefore\ y^2 = C\left(x^2 - \frac{B}{1 + AC}\right)$$

the complete integral, C being the arbitrary constant.

Also taking the other factor, we have

$$y^2 = u\,(x^2 - x\sqrt{B}) = \frac{1}{A}(x - \sqrt{B})\,(\sqrt{B} - x),$$

$$\text{or}\quad (y\sqrt{A})^2 + (x - \sqrt{B})^2 = 0,$$

the singular solution, which resolves itself into the two

$$y = 0,\ x = \pm\sqrt{B} = \pm a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

and indicates the umbilici of the surface (Prob. IV.), the normals at which, as we know, are intersected by the normals drawn from consecutive points in every direction on the surface.

In order to determine the constant in the complete integral, let x', y', be the given co-ordinates of the point through which the lines of curvature pass,

$$\therefore\ y'^2 = C\left(x'^2 - \frac{B}{1 + AC}\right)\ \text{or}\ C^2 - C\,\frac{Ay'^2 + B - x'^2}{Ax'^2} = \frac{y'^2}{Ax'^2},$$

$$\therefore\ C = \frac{Ay'^2 + B - x'^2}{2Ax'^2} \pm \frac{1}{2Ax'^2}\sqrt{(Ay'^2 + B - x'^2)^2 + 4Ax'^2y'^2},$$

$$\text{and}\ 1 + AC = \frac{Ay'^2 + B + x'^2}{2x'^2} \pm \frac{1}{2x'^2}\sqrt{(Ay'^2 + B + x'^2)^2 - 4Bx'^2}.$$

Hence, we see that C has two real values one $+$, and the other $-$, and that $1 + AC$ is always $+$; hence the complete integral represents an ellipse when C is $-$, and a hyperbola, when C is $+$; and therefore the projections of the lines of curvature passing through any point on the surface of an ellipsoid, upon the principal plane containing the greatest and mean axes, are an ellipse and hyperbola.

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In the first case, if m and n be the semi-axes of the ellipse,

$$\frac{n^2}{m^2} = C, \quad m^2 = \frac{B}{1 - AC},$$

therefore m and n are connected by the equation

$$\frac{m^2}{B} - \frac{An^2}{B} = 1;$$

in the second case, if m and n be the semi-axes of the hyperbola,

$$\frac{n^2}{m^2} = C, \quad m^2 = \frac{B}{1 + AC}, \quad \therefore \frac{m^2}{B} + \frac{An^2}{B} = 1.$$

Hence the semi-axes for each ellipse in the projection are the co-ordinates of a point in the same given hyperbola, and for each hyperbola, the semi-axes are the co-ordinates of a point in the same given ellipse; these are called the *auxiliary* ellipse and hyperbola; they are concentric with the ellipsoid, and their semi-axes, (which are the same for both curves, and coincide in direction with those of the ellipsoid) have the following values

$$\tfrac{1}{2} \text{ axis major} = \sqrt{B} = a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

$$\tfrac{1}{2} \text{ axis minor} = \sqrt{\frac{B}{A}} = b\sqrt{\frac{a^2 - b^2}{b^2 - c^2}}.$$

Hence we have the following construction for the projection, upon the plane containing the greatest and mean axes, of the lines of curvature of an ellipsoid. Let ACB (fig. 68.) be this plane (that of xy), and with the axes found above construct the auxiliary ellipse and hyperbola GO, HO. From any point I in the hyperbola, and i in the ellipse, let fall perpendiculars on the axes CA, CB; and with semi-axes CN, CM, construct the ellipse MN, this is the projection of a line of the first curvature; and in the same manner the projections of any number of lines of the first curvature may be constructed. Again with semi-axes mi, ni, construct the hyperbola nK, this is the projection

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of a line of the second curvature; and similarly any number of projections of lines of the second curvature may be constructed. As the points I and i approach O and G, the ellipse and hyperbola continually approach the lines CO, CB, and are at last confounded with them; therefore CO, CB are the projections of lines of curvature, and consequently the curves in which the surface is cut by the principal planes of xz, yz, are lines of curvature; also if in the equation to the auxiliary hyperbola we make $m = a$, we find $n = b$, therefore the ellipse AB is included in the construction, and is itself a line of curvature. Hence the intersections of the surface with its three principal planes are all lines of curvature. Since $CO = a\sqrt{\dfrac{a^2-b^2}{a^2-c^2}}$ and is therefore $< a$, the point O falls within the ellipse AB; it is the projection of an umbilicus of the surface (Art. 235.) round which the lines of both curvatures are symmetrically disposed, and towards which they all turn their concavities.

VI. To construct the projections of the lines of curvature of an ellipsoid upon the plane containing the greatest and least axes.

If we suppose the plane of projection, that is, the plane of xy, to contain the greatest and least axes of the ellipsoid, we shall find the construction easier. In that case α, γ, β are in order of magnitude, and $\therefore A = -\dfrac{\alpha(\gamma-\beta)}{\beta(\alpha-\gamma)} = -A_1$ suppose; B still remains positive. Hence

$$y^2 = C\left(x^2 - \frac{B}{1-A_1C}\right),$$

$$C = \frac{1}{2A_1x'^2}\left\{A_1y'^2 - B + x'^2 \pm \sqrt{(A_1y'^2 - B + x'^2)^2 - 4A_1x'^2y'^2}\right\}.$$

Hence both values of C have the same sign, and are both negative if $\dfrac{A_1y'^2}{B} + \dfrac{x'^2}{B}$ be less than 1, *i. e.* if

$$\frac{\gamma-\beta}{\alpha-\beta}\frac{y'^2}{\beta} + \frac{\alpha-\gamma}{\alpha-\beta}\frac{x'^2}{\alpha} \text{ be less than } 1,$$

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which must always be the case, because the given point being in the surface of the ellipsoid, we must have

$$\frac{y'^2}{\beta} + \frac{x'^2}{\alpha} < 1.$$

Hence both projections are ellipses; and if m and n be the semi-axes of any ellipse, we have

$$\frac{n^2}{m^2} = C, \quad m^2 = \frac{B}{1 + A_1 C}, \quad \therefore \quad \frac{m^2}{B} + \frac{A_1 n^2}{B} = 1,$$

which is the equation to an ellipse whose semi-axes are

$$\sqrt{B} = a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \text{ and } \sqrt{\frac{B}{A_1}} = b\sqrt{\frac{a^2 - b^2}{c^2 - b^2}}.$$

In this case, therefore, the semi-axes of each ellipse of the projection are the co-ordinates of a point taken on another given ellipse, which is the same for all.

Hence let BA, (fig. 69), be the principal section of the ellipsoid containing the greatest and least axes; and concentric with it, describe the auxiliary ellipse YX with the semi-axes just found, and which we observe are greater than the corresponding semi-axes of the ellipsoid a and b. From any points I, i, let fall perpendiculars on OX, OY, and with these as semi-axes construct the ellipses MN, mn; then MN, mn, are the projections of lines of curvature. If in the equation to the auxiliary ellipse we make $m = a$, we find $n = b$; therefore the principal section is itself included in the construction, and if from the points A and B we draw tangents meeting the auxiliary ellipse in D, the perpendiculars dropped from every point in DY will serve to construct the projections of lines of the first curvature, and perpendiculars dropped from the points in DX will serve for the construction of the projections of lines of the second curvature. It is not difficult to shew that if XY be joined, all the ellipses in the projection are touched by it, and consequently they are all inscribed in the same equilateral parallelogram.

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This however must appear from the singular solution of the differential equation to the lines of curvature, for that represents a curve which always touches and envelopes the curves comprized in the complete integral.

Now the factor $x^2 - \frac{B}{(1+Au)^2}$ (p. 260) which in this case becomes $x^2 - \frac{B}{(1-A_1u)^2}$, put equal to zero, gives a value of u which substituted in the equation

$$y^2 = x^2u - \frac{Bu}{1-A_1u} \text{ gives } y^2 = \frac{1}{A_1}(x - \sqrt{B})^2;$$

and this equation represents the four chords which pass through the vertices of the auxiliary ellipse, and of which YX is one touching AB at an umbilicus of the surface.

Lastly, let the plane of projection contain the mean and least axes, *i. e.* let γ, β, α be in order of magnitude, then $B = \alpha\frac{\beta-\alpha}{\gamma-\alpha}$, $A = \frac{\alpha}{\beta}\frac{\gamma-\beta}{\gamma-\alpha}$, which are both positive; this is consequently resolved into the first case.

The conclusion from the whole investigation is, that if the two lines of curvature which pass through any point on the surface of an ellipsoid be projected upon the principal planes, the projections will be an ellipse and hyperbola, or two ellipses, according as the plane of projection does or does not contain the mean axis of the ellipsoid.

VII. To determine the radii of curvature of an ellipsoid.

Let the equation be $\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1$;

$$\therefore p = -\frac{\gamma}{z}\frac{x}{\alpha}, \quad q = -\frac{\gamma}{z}\frac{y}{\beta}, \quad r = -\frac{\gamma^2}{\alpha\beta z^3}(\beta - y^2),$$

$$s = -\frac{\gamma^2}{\alpha\beta z^3}xy, \quad t = -\frac{\gamma^2}{\alpha\beta z^3}(\alpha - x^2);$$

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$$\therefore\ (1+p^2)\,t = -\frac{\gamma^2}{\alpha\beta z^3}\left\{a - x^2 + \left(\frac{\gamma}{z}\right)^2\left(\frac{x}{\alpha}\right)^2(a-x^2)\right\},$$

$$-2pqs = -\frac{\gamma^2}{\alpha\beta z^3}\left\{-\left(\frac{\gamma}{z}\right)^2\frac{2x^2y^2}{\alpha\beta}\right\},$$

$$(1+q^2)\,r = -\frac{\gamma^2}{\alpha\beta z^3}\left\{\beta - y^2 + \left(\frac{\gamma}{z}\right)^2\left(\frac{y}{\beta}\right)^2(\beta-y^2)\right\}.$$

But in taking the sum of these three equations, the coefficient of $\left(\frac{\gamma}{z}\right)^2$ within the brackets is

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}-\left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)^2 = \left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}\right)\left(1-\frac{x^2}{\alpha}-\frac{y^2}{\beta}\right) = (\gamma - z^2)\left(\frac{z}{\gamma}\right)^2;$$

$$\therefore\ (1+p^2)\,t - 2pqs + (1+q^2)\,r =$$

$$-\frac{\gamma^2}{\alpha\beta z^3}\left\{\alpha+\beta+\gamma-(x^2+y^2+z^2)\right\}.$$

$$\text{Also } rt - s^2 = \left(\frac{\gamma^2}{\alpha\beta z^3}\right)^2\left\{(\beta-y^2)(\alpha-x^2) - x^2y^2\right\}$$

$$= \alpha\beta\left(\frac{\gamma^2}{\alpha\beta z^3}\right)^2\frac{z^2}{\gamma} = \frac{\gamma^2}{\alpha\beta z^3}\frac{\gamma}{z},$$

$$\text{and } 1+p^2+q^2 = \frac{\gamma^2}{z^2}\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right).$$

$$\text{Let } A^2 = \alpha+\beta+\gamma = a^2+b^2+c^2,$$

D = the distance of the point in question from the center,

$$\therefore\ D^2 = x^2+y^2+z^2;$$

P = the perpendicular from the center on the tangent plane,

$$\therefore\ \frac{1}{P^2} = \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2},$$

as may be easily shewn from Prob. 3, Art. 120; therefore, by substitution in the equation of Art. 233, and reduction,

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$$\left(\frac{\gamma}{z}\right)^2 (z'-z)^2 + \frac{\gamma}{z}(A^2-D^2)(z'-z) + \frac{\alpha\beta\gamma}{P^2} = 0.$$

But $R = -(z'-z)\sqrt{1+p^2+q^2} = -\dfrac{(z'-z)\gamma}{Pz}$,

or $\dfrac{(z'-z)\gamma}{z} = -PR$;

$$\therefore\ R^2 - \frac{A^2-D^2}{P}R + \frac{\alpha\beta\gamma}{P^4} = 0.$$

Let R', R'' denote the values of R in this equation, then

$$R'.R'' = \left(\frac{abc}{P^2}\right)^2, \quad R'+R'' = \frac{A^2-D^2}{P},$$

which is the simplest form to which the result can be reduced; the first equation shews that at points where the tangent planes are equidistant from the center, the product of the radii is constant; and the second that at points which are themselves equidistant from the center, the sum of the radii varies inversely as the perpendicular on the tangent plane. Hence the volume of the ellipsoid may be expressed by the two radii of curvature at any point, and the perpendicular on the tangent plane, for it

$$= \frac{4\pi abc}{3} = \frac{4\pi}{3}P^2.\sqrt{R'R''}.$$

Cor. 1. Since the equations of condition which characterize umbilici always make the differential equation of the lines of curvature identical, and since that equation in the present case is (p. 260)

$$Axy(d_x y)^2 + (x^2 - Ay^2 - B)d_x y - xy = 0 \ldots\ldots (1),$$

we must have

$$xy = 0, \quad x^2 - Ay^2 - B = 0;$$

but we cannot suppose $x = 0$, for that gives y^2 equal to a negative quantity, therefore we must have

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$$y = 0, \quad x^2 = B, \quad \text{or } x = \pm a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}},$$

which equations indicate four points placed symmetrically in the four angles of the principal section containing the greatest and least axes, and of two of which O (fig. 68.) is the projection. Also the radius of curvature in this case (since the two radii become equal to one another)

$$= \frac{abc}{P^2} = abc\left(\frac{x^2}{a^4} + \frac{z^2}{c^4}\right) = \frac{b^3}{ac}.$$

We have already found the co-ordinates of the umbilici by considering that they are situated at the extremities of the diameters which pass through the centers of all the circular sections of an ellipsoid; also the radius of curvature of the principal section corresponding to these co-ordinates, will be that at the umbilicus; from which consideration the radius of curvature at the umbilicus of an ellipsoid may be obtained, without deducing it from the general expression.

Cor. 2. In constructing the projections of the lines of curvature of an ellipsoid upon the plane containing the greatest and the mean axis, we saw that for each of the umbilici the lines of curvature passing through it were reduced to a single one, viz. the vertical ellipse projected into COA. This also appears from the differential equation to the lines of curvature; for since $d_x y$, for the particular values of x and y belonging to an umbilicus, assumes the form $\frac{0}{0}$, we must, to obtain its true value in that case, recur to the next derived equation; this gives, not writing down the terms which involve $d_x^2 y$,

$$Ax(d_x y)^3 - Ay(d_x y)^2 + x d_x y - y = 0;$$

therefore at an umbilicus when $y = 0$, $x = \pm\sqrt{B}$, this equation is decomposed into

$$d_x y = 0, \quad A(d_x y)^2 + 1 = 0;$$

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therefore $d_x y = 0$ is the only admissible value, which when integrated and corrected so as to pass through an umbilicus, gives $y = 0$, and indicates the vertical ellipse which is projected into *COA*. Several of the above remarks are due to C. F. A. Leroy, who on this subject has made valuable additions to the well known labours of Monge and Dupin.

VIII. To find the radii of curvature at any point of a paraboloid.

Let the equation to the surface be

$$\frac{x^2}{a} + \frac{y^2}{b} + 2z = 0;$$

then it may be shewn, as in the preceding example, that the maximum and minimum radii of curvature at a point xyz, are

$$= \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1} \left\{ \frac{a + b - 2z}{2} \right.$$

$$\left. \pm \sqrt{\left(\frac{a + b - 2z}{2}\right)^2 - ab\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1\right)} \right\}.$$

Hence if the radii be projected on the axis of z, the sum of the projections $= a + b - 2z$.

IX. In surfaces of the second order, the two curvatures are in the same direction at every point of the same surface, or they are in opposite directions at every point.

The curvatures are in the same or opposite directions at any point of a surface, according as $rt - s^2$ for that point is positive or negative. But, substituting for r, s, t their values from

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1,$$

we find

$$rt - s^2 = \frac{\gamma^4}{z^4} \cdot \frac{1}{\alpha\beta\gamma},$$

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a quantity which can never change its sign by the variation of x, y, z, and will always have the same sign as $\alpha\beta\gamma$; therefore for the ellipsoid and hyperboloid of two sheets, it will be positive, or the two curvatures at every point of the surface will be in the same direction; and for the hyperboloid of one sheet it will be negative, or the two curvatures at every point will be in opposite directions. Similarly, for the elliptic and hyperbolic paraboloids the curvatures are respectively in the same, and opposite directions, at every point.

The following Problem, though unconnected with the subject of the section, is added for the sake of the results to which it leads, especially the equation to the Wave-surface in the last example; in that example the method of elimination is taken with some alterations from a recent Paper in the *Cambridge Philosophical Transactions.*

X. To determine the envelope, when the equation to the series of surfaces contains two independent parameters.

Suppose the equation $u = 0$ to contain two parameters α and β independent of one another; then the equation to the surface which differs insensibly from that represented by $u = 0$, will be

$$u + d_\alpha u \,.\, \delta\alpha + d_\beta u \,.\, \delta\beta + \&c. = 0;$$

the co-ordinates of the points in which these surfaces intersect must satisfy their two equations, that is, they must satisfy

$$u = 0, \quad d_\alpha u \,.\, \delta\alpha + d_\beta u \,.\, \delta\beta + \&c. = 0,$$

or, if the surfaces be consecutive,

$$u = 0, \quad d_\alpha u + m \,.\, d_\beta u = 0,$$

whence $m =$ the limit of $\dfrac{\delta\beta}{\delta\alpha}$; but, since m may have any value, this latter equation resolves itself into

$$d_\alpha u = 0, \quad d_\beta u = 0;$$

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these then, together with $u = 0$, are the three equations for determining the curve of intersection of two consecutive surfaces; from whence we may obtain its two equations, involving only one of the parameters, and finally, by eliminating that parameter, we may obtain the equation to the surface generated by the perpetual intersections of the first series of surfaces.

Ex. 1. To find the equation to the surface touched by a series of planes so drawn, that the product of the perpendiculars let fall upon any one from two fixed points is invariable.

Let the line joining the two points be the axis of x, $2a$ its length, and the origin in its middle point; $z = Ax + By + c$ the equation to one of the planes; then the lengths of the perpendiculars are

$$\frac{c + Aa}{\sqrt{1 + A^2 + B^2}}, \qquad \frac{c - Aa}{\sqrt{1 + A^2 + B^2}};$$

$$\therefore \frac{c^2 - A^2 a^2}{1 + A^2 + B^2} = b^2 \text{ suppose, or } c^2 = b^2(1 + A^2 + B^2) + A^2 a^2.$$

Hence, differentiating the equation to the plane, successively with respect to A and B, considering c as a function of A and B by virtue of the above equation, we have

$$\therefore 0 = x + (a^2 + b^2)\frac{A}{c}, \quad \text{or } x = -(a^2 + b^2)\frac{A}{c},$$

$$0 = y + b^2\frac{B}{c}, \qquad \text{or } y = -b^2\frac{B}{c};$$

$$\therefore z = -(a^2 + b^2)\frac{A^2}{c} - \frac{b^2 B^2}{c} + c = \frac{-(a^2 + b^2)A^2 - b^2 B^2 + c^2}{c} = \frac{b^2}{c};$$

$$\therefore \frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = \frac{(a^2 + b^2)A^2 + b^2 B^2 + b^2}{c^2} = 1,$$

the equation to a prolate spheroid, whose axis coincides with the line joining the two given points.

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Ex. 2. Let the equation to a surface be $\frac{x^m}{a^m}+\frac{y^m}{b^m}+\frac{z^m}{c^m}=1$, to find the equation to the surface which it touches in all the positions it can assume, subject to the condition $a^n+b^n+c^n=k^n$, a constant quantity.

Differentiating the equation to the surface successively with respect to a and b, regarding c as a function of a and b, by virtue of the equation of condition, we have

$$-\frac{x^m}{a^{m+1}}+\frac{z^m}{c^{m+1}}\cdot\frac{a^{n-1}}{c^{n-1}}=0, \quad \text{or} \quad \frac{x^m}{a^m}=\frac{z^m}{c^m}\cdot\frac{a^n}{c^n};$$

$$-\frac{y^m}{b^{m+1}}+\frac{z^m}{c^{m+1}}\cdot\frac{b^{n-1}}{c^{n-1}}=0, \quad \text{or} \quad \frac{y^m}{b^m}=\frac{z^m}{c^m}\cdot\frac{b^n}{c^n};$$

$$\therefore \frac{x^m}{a^m}+\frac{y^m}{b^m}+\frac{z^m}{c^m}=\frac{z^m}{c^m}\cdot\frac{a^n+b^n+c^n}{c^n}=\frac{z^m}{c^m}\cdot\frac{k^n}{c^n}=1; \quad \therefore \frac{z^m}{k^m}=\frac{c^{m+n}}{k^{m+n}};$$

$$\text{or } \left(\frac{z}{k}\right)^{\frac{mn}{m+n}}=\frac{c^n}{k^n}, \quad \text{similarly } \left(\frac{x}{k}\right)^{\frac{mn}{m+n}}=\frac{a^n}{k^n}, \quad \left(\frac{y}{k}\right)^{\frac{mn}{m+n}}=\frac{b^n}{k^n};$$

$$\therefore x^{\frac{mn}{m+n}}+y^{\frac{mn}{m+n}}+z^{\frac{mn}{m+n}}=k^{\frac{mn}{m+n}}, \text{ the equation required.}$$

If $m=n=1$, the equation becomes $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{k}$, and agrees with Prob. 7. Art. 120; if $m=n=2$, it is $x+y+z=k$, which shews that if a series of surfaces of the second order have the sum of the squares of their axes constant, they are all touched by a plane.

Similarly, if the equation of condition be $abc=k^3$, it may be shewn that the equation to the enveloping surface is

$$xyz=\left(\frac{k^m}{3}\right)^{\frac{3}{m}};$$

and if $m=1$, we find $xyz=\frac{k^3}{27}$ for the equation to the surface touched by a series of planes which form with the co-ordinate planes a pyramid of constant volume. The

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surface represented by the equation xyz = a constant, has also the property that the tangent plane at any point x, y, z, forms with the co-ordinate planes a pyramid of smaller volume than any other plane drawn through the point of contact, its equation being

$$\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z} = 3;$$

it may easily be shewn that this is the equation to the tangent plane of any surface whatever, which forms with the co-ordinate planes the least possible pyramid, x, y, z, being the co-ordinates of the point of contact.

Ex. 3. To find the equation to the surface touched by a series of planes which cut off from a given right cone, an oblique cone, such that its volume is constant, or such that the transverse axis of its elliptic base is constant.

If the volume of the oblique cone be constant, then by a property of conic sections the transverse axis of the elliptic base is also constant, suppose it $\therefore = 2c$, and let β = semi-vertical angle of the cone; then it will be found that the equation to the touched surface is

$$z^2 = (\cot \beta)^2 (x^2 + y^2 + c^2),$$

the origin being at the vertex; which belongs to an hyperboloid of revolution about the axis of the cone, and to which the surface of the cone is an asymptote, the $\frac{1}{2}$ axes of the generating hyperbola being c and $c \cot \beta$; also the point of contact is the center of the elliptic section.

In like manner if a series of planes cut off from a paraboloid of revolution a segment of constant volume, the surface touched by them is a similar and equal paraboloid about the same axis.

Ex. 4. To find the equation to the surface which is always touched by a plane, whose equation is

$$mx + ny + lz = v \ldots\ldots (1),$$

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m and n being independent parameters, and l and v functions of m and n determined by the equations

$$m^2 + n^2 + l^2 = 1 \ldots\ldots (2),$$

$$\frac{m^2}{v^2 - a^2} + \frac{n^2}{v^2 - b^2} + \frac{l^2}{v^2 - c^2} = 0 \ldots\ldots (3).$$

Differentiating (1) with respect to m and n, and taking account of (2), we have

$$x + z\left(-\frac{m}{l}\right) = d_m v, \qquad y + z\left(-\frac{n}{l}\right) = d_n v.$$

But differentiating (3) with respect to m, and putting for $d_m v$ the value found above, we have, making

$$\frac{m^2}{(v^2 - a^2)^2} + \frac{n^2}{(v^2 - b^2)^2} + \frac{l^2}{(v^2 - c^2)^2} = A,$$

$$\frac{1}{v^2 - a^2} - \frac{1}{v^2 - c^2} = A\left(\frac{x}{m} - \frac{z}{l}\right) v;$$

similarly, $$\frac{1}{v^2 - b^2} - \frac{1}{v^2 - c^2} = A\left(\frac{y}{n} - \frac{z}{l}\right) v;$$

therefore, multiplying by m^2 and n^2, and adding,

$$\frac{m^2}{v^2 - a^2} + \frac{n^2}{v^2 - b^2} + \frac{l^2 - 1}{v^2 - c^2} = A\left(mx + ny + \frac{l^2 - 1}{l} z\right) v,$$

$$\text{or } \frac{1}{v^2 - c^2} = A\left(\frac{z}{l} - v\right) v \ldots\ldots (4) \quad \text{by (1) and (3)};$$

$$\therefore \frac{1}{v^2 - a^2} = A\left(\frac{x}{m} - v\right) v,$$

$$\frac{1}{v^2 - b^2} = A\left(\frac{y}{n} - v\right) v.$$

Therefore multiplying the squares of these three equations by l^2, m^2, n^2, and adding,

$$A = A^2(x^2 + y^2 + z^2 + v^2 - 2v^2)v^2 \quad \text{by (1) and (2)},$$

$$\text{or } 1 = A(r^2 - v^2)v^2, \quad \text{making } r^2 = x^2 + y^2 + z^2.$$

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Hence (4) becomes

$$\frac{v^2-c^2}{r^2-v^2}=\frac{lv}{\varkappa-lv},\quad\text{or}\quad\frac{c^2-v^2}{v^2}=\frac{lv^2-lr^2}{\varkappa v-lv^2};$$

and adding unity to each side of these equations we find

$$\frac{r^2-c^2}{r^2-v^2}=\frac{\varkappa}{\varkappa-lv},\quad\frac{c^2}{v^2}=\frac{\varkappa v-lr^2}{\varkappa v-lv^2};$$

therefore eliminating $\varkappa-lv$, we have

$$\frac{c^2\varkappa}{r^2-c^2}=\frac{v(\varkappa v-lr^2)}{r^2-v^2};$$

similarly

$$\frac{a^2x}{r^2-a^2}=\frac{v(xv-mr^2)}{r^2-v^2};$$

$$\frac{b^2y}{r^2-b^2}=\frac{v(yv-nr^2)}{r^2-v^2};$$

therefore multiplying by $\varkappa$, x, y, respectively, and adding, we have the required equation to the surface,

$$\frac{c^2\varkappa^2}{r^2-c^2}+\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}=\frac{v}{r^2-v^2}(r^2v-r^2v)=0.$$

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ERRATA.

Page 10,	line 12,	*for*	fig. 6	*read*	fig. 6 bis.
..... 46,	last line	....	$\frac{x^2}{l^2}$	.....	$\frac{l^2}{c^2}$.
..... 64,	line 22,	....	Art. 25	.....	Art. 27.
..... 76,	 7,	....	b^2	.....	y^2.
.....104,	 6,	....	$=\frac{z^2}{c^2}+1$	.....	$+\frac{z^2}{c^2}=1$.

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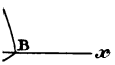
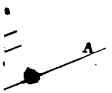
Plate 1.





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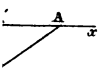

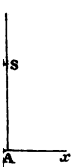

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TWO GEOMETRICAL MEMOIRS

ON THE

GENERAL PROPERTIES

OF

CONES OF THE SECOND DEGREE

AND ON THE

SPHERICAL CONICS,

BY

M. CHASLES.

TRANSLATED FROM THE FRENCH, WITH NOTES AND ADDITIONS,

AND

AN APPENDIX

ON THE

APPLICATION OF ANALYSIS TO SPHERICAL GEOMETRY,

BY THE

REV. CHARLES GRAVES, A.M., M.R.I.A.,

FELLOW AND TUTOR OF TRINITY COLLEGE, DUBLIN.

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PREFACE.

THE two Memoirs by M. Chasles, "On the Properties of Cones of the second Degree," and "On the Spherical Conics," of which a translation is now presented to English geometers, were printed in the sixth volume of the Transactions of the Royal Academy of Brussels. Whether they are considered merely as exercises of pure geometry, exhibiting its elegance and power in a remarkable degree, or as a rich and early contribution to the theory of spherical curves, they possess strong claims on the attention of mathematicians.

But, published as they were, they remained unseen by the greater number of our geometers, so that it appeared highly desirable to make them more generally known by means of a reprint or translation; more especially as there is no detached work in the English language treating of the same subject. Most readers would doubtless prefer to see M. Chasles' Memoirs reprinted in their original language; but the small additional trouble which the translation of them has cost, will not have been vainly incurred, if a few readers

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benefit by it who are not perfectly familiar with the French idiom.

The matter contained in the Notes and Additions by the Translator is for the most part original. Some theorems relating to the anharmonic function of four points and to the involution of six points, have been borrowed from the notes to M. Chasles' admirable Histoire de la Geometrie.

In the Appendix will be found the outline of a system of analytic geometry, intended to accomplish for the surface of the sphere what the method of rectilinear coordinates has already effected for the plane. The Author has barely traced this outline: any reader tolerably conversant with the common processes of algebraic geometry will be able to fill it up; and none will complain of the incompleteness of a sketch which comprises in five and twenty pages the leading principles of the algebraic geometry of the sphere, and their application to the spherical conics. Those who are anxious to pursue this subject farther, will find formulæ for the transformation of spherical coordinates, and a discussion of the general equation of the second degree, in a paper which the Author had the honour of laying before the Royal Irish Academy on the 28th of June, 1841.

The twelfth volume of the Transactions of the Royal Society of Edinburgh contains two elaborate papers on the use of spherical coordinates, by Mr. S. T. Davies. The method which he employs leads him necessarily to unsymmetrical results, just such as we should meet with, if, in the analytic geometry of the plane, we restricted

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ourselves to the use of polar coordinates; and, in fact, Mr. Davies' method ought to be called that of *spherical polar coordinates.* But it may often be used with advantage; a few of the formulæ most necessary in its application are therefore given in the thirteenth section of the Appendix.

After the Author had constructed the theory of spherical coordinates, which he now lays before the reader, his attention was directed, by a note attached to one of Mr. Davies' papers, to articles published by Professor Gudermann, of Cleves, in Crelle's Journal of Pure and Applied Mathematics. From them he learned, to his regret, he must own, that he had been anticipated in the choice of his coordinates by Professor Gudermann, who has employed them successfully in investigating general properties of the spherical conics. However, as the Author has not yet been able to procure the work on Analytic Spherics, written by the Professor, and referred to in the papers inserted in Crelle's Journal, he is ignorant as to whether he has been also anticipated in the use of the differential calculus in discussing curves represented by an equation between spherical coordinates, and in the invention of general formulæ for the transformation of such coordinates. These steps being made, the theory is complete, and nothing remains but to apply it.

The Author had intended to add a second Appendix, containing those theorems relative to the plane conic sections, which may be deduced from the properties of the spherical conics stated in his Notes and Additions,

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ON

THE GENERAL PROPERTIES

OF

CONES OF THE SECOND DEGREE.

In a preceding Memoir on the general properties of the surfaces of revolution of the second degree, inserted by order of the Royal Academy in the collection of its memoirs, we demonstrated various properties of the cones of the second degree, which presented themselves as immediate consequences of the principles which we had founded upon the polar transformations of these cones and of the conic sections.

We now propose to discuss, by a direct method, some of the general properties of the cones of the second degree. We might have continued to employ the theory of polar transformations; but there is a more simple mode of proceeding, altogether independent of this theory. This mode is purely geometrical, requiring only a knowledge of the most elementary properties of the circle.

The only properties of the cones of the second degree which we shall assume as known are the two following:

Prop. 1. (1) "In every cone of the second degree there are three rectan-
Prop. 2. gular conjugate axes. (2) There are also two series of circular sections, situated in planes parallel to two fixed planes."

This second proposition, which we receive without demonstration, as it is to be found in the elementary treatises on surfaces of the second degree, is the basis of our entire work;* and the numerous theorems to which we shall be led will all be deductions from this principle; and easy deductions, inasmuch as they will appear to result from elementary properties of the circle; so that we might say that our various theorems

* This general property of the cones of the second degree is due, I believe, to Descartes, who demonstrated it by the new principles of his geometry. His proof is to be found in the 6th vol. of his *Lettres* (12mo. ed. 1724, 1725.)

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are, as it were, inscribed upon the planes of the circular sections of the cones of the second degree.

These theorems might also be demonstrated by algebraic analysis; but this method, which in general offers so great advantages, loses them all in this case, since it often requires very tedious calculations, and exhibits no connexion between the different propositions; so that it is only useful in verifying those which are already known, or whose truth has been otherwise suggested as probable.

SECTION I.

PRELIMINARY CONSIDERATIONS.

1. A cone of the second degree is one which has a conic section for its base.

Let there be two tangent planes; their traces on the plane of the conic section will be two tangents to this curve; the plane containing the two sides of the cone along which it is touched by the tangent planes, will meet the plane of this conic section in the chord which joins the points of contact of the two tangents; this chord is the polar, with relation to the conic section, of the point of concourse of the two tangents: hence we say that the plane of the two sides, along which the two tangent planes touch the cone, is the *polar plane*, with relation to the cone, of the right line of intersection of these two tangent planes; and this right line is called the *polar* of the plane of these two sides of contact.

It is well known that, in the conic section which is the base of the cone, all the right lines drawn through the same point have their poles upon the polar of this point; it follows, therefore, that

In a cone of the second degree, all the planes passing through the same axis have their* polars *situated in the* polar plane *of this axis.*

Hence, if, in the polar plane of any axis, we draw arbitrarily a second axis, the polar plane of this latter will pass through the first axis; and so these two axes are called *conjugate:* their polar planes are also said to be mutually *conjugate;* so that *two diametral planes of a cone of the second degree are* conjugate, *when the polar of one of them is in the other plane.*

2. Since every transversal plane, which cuts the cone in a conic section, cuts the polar plane of any axis in a right line

* We apply the name "axis" to every right line passing through the vertex of the cone.

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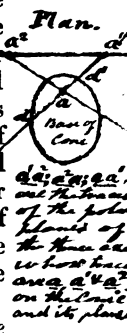
which is the polar, with relation to the conic section, of the point where this plane meets this axis, it follows that if this transversal plane be parallel to the polar plane of the axis, it will cut the cone in a conic section whose centre will be the point where it cuts this axis, since the polar of this point will be at an infinite distance.

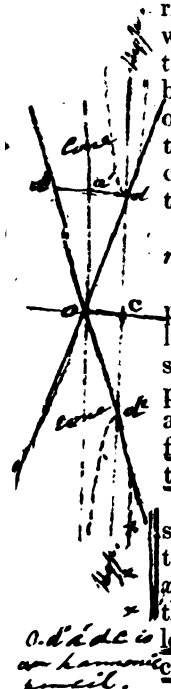
Hence, *the polar plane of any axis of a cone is such that every plane which is parallel to it cuts the cone in a conic section whose centre is upon this axis;* and conversely, *the polar axis of a plane is the geometrical locus of the centres of the sections made in the cone by planes parallel to this plane.*

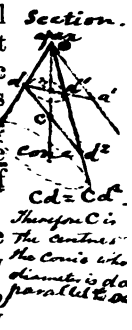
3. Through the vertex of a cone of the second degree, we might draw, in an infinite number of different ways, three right lines such that each of them should be the polar of the plane of the two others. For, after having taken the polar plane of one right line, let us take the polar plane of a second right line drawn in this first polar plane; these two planes will intersect in a third right line, which will be the polar of the plane of the two first; so that these three right lines will be so related to each other that each of them will be the polar of the plane of the two others; and consequently, each of them is the locus of the centres of the sections made in the cone by planes parallel to the plane of the two others: these three right lines form a *system of conjugate axes.*



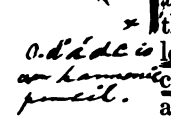
Hence, *in every cone of the second degree there is an infinite number of systems of three conjugate axes.*

4. It is clear that three conjugate axes meet a transversal plane, in three points of which each is the pole of the right line which joins the two others, with relation to the conic section in which this plane cuts the cone; if, therefore, this plane be parallel to the plane of two of the three conjugate axes, it will cut the two other faces of the trihedral angle formed by these three axes, along two conjugate diameters of the conic section in which this plane cuts the cone.





These properties of three conjugate axes of a cone of the second degree shew that it is very useful in analytic geometry to take three conjugate axes for the three axes of coordinates, x, y, z, because then the equation of the cone contains only the squares of the three coordinates. For every plane parallel to the plane of two of these axes will cut the cone in a conic section which will have its centre upon the third axis, and two conjugate diameters respectively parallel to the two first axes; consequently, the equation of the projection of this conic section upon the plane of the two first axes will be referred to two conjugate diameters, and will, therefore, con-



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tain only the squares of the two coordinates; which proves that the equation of the cone itself contains only the squares of the three coordinates.

5. It is known that, amongst all the systems of conjugate axes belonging to a cone of the second degree, there is one in which the three axes are rectangular. Hence, every plane perpendicular to one of these axes cuts the cone in a conic section which has its centre upon this axis, and its principal diameters parallel to the two other axes.

One of these axes lies in the interior of the cone, and every plane which is perpendicular to it cuts the cone in an ellipse; the two other axes are outside the cone, and every plane perpendicular to either of them cuts the cone in a hyperbola whose asymptotes are parallel to the two sides of the cone, which are parallel to the cutting plane.

Of the three rectangular conjugate axes, the one which lies in the interior of the cone will henceforward be designated as the *principal axis of the cone;* that which is parallel to the major axis of the ellipse, in which the cone is cut by a plane perpendicular to the principal axis, will be called the *major axis* of the cone; and the one which is parallel to the minor axis of this ellipse, will be denominated the *minor axis* of the cone.

The plane which contains the principal axis and the major axis will be the plane of the *greatest section* of the cone; the plane which contains the principal axis and the minor axis will be the plane of the *least section* of the cone; and finally, the plane which contains the major and the minor axes will be the *principal plane* of the cone.

It is manifest, that the principal plane does not cut the cone along any side; that, among all planes which can be drawn through the principal axis, the plane of the greatest section is that which cuts the cone along the two sides which contain between them the greatest angle; and that the plane of the least section is that which cuts it along the two sides which contain between them the least angle.

The tangent planes to the cone passing through the axis major, touch it along the two sides contained in the plane of the least section; the tangent planes passing through the minor axis touch it along the two sides contained in the plane of the greatest section; and finally, through the principal axis no tangent plane can be drawn to the cone.

It is further to be remarked, that every plane passing through the principal axis cuts the cone along two sides, which make equal angles with this axis; and that if a plane,

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passing through the major or minor axis, cuts the cone, the two sides of intersection will also make equal angles with this axis.

If the section of the cone, made by a plane perpendicular to its principal axis, be a circle, instead of being an ellipse, all the sides of the cone will make the same angle with this principal axis, and the cone will be one of revolution.

6. Let there be a cone of the second degree; through its vertex let us draw right lines perpendicular to its tangent planes, they will form a second cone, which will be of the second degree; since every plane passing through its vertex can only cut it along two sides; for if a plane were to cut it along three sides, these sides would be perpendicular to three planes touching the proposed cone, which planes would pass through the same right line perpendicular to the plane of the three sides; which is impossible, since through a right line only two tangent planes can be drawn to a cone of the second degree. Consequently, the second cone is of the second degree.

Supplementary Cones.

Two adjacent sides of this cone, that is, two sides infinitely near, are perpendicular to two adjacent planes touching the proposed cone; the plane of these two sides is perpendicular to the right line of intersection of the two adjacent tangent planes; that is to say, the tangent planes to the second cone are perpendicular to the sides of the first; we have, therefore, the following theorem:

The right lines drawn through a fixed point perpendicular to the tangent planes to a cone of the second degree, form a second cone of the second degree;

And the tangent planes to this second cone are perpendicular to the sides of the first.

7. Hence, to each side A in the first of the two cones *corresponds* a plane P touching the second cone, which plane is perpendicular to this side A.

This plane P touches the second cone exactly along the side which corresponds to the plane touching the first cone along the side A. For the side of contact of the plane P and of the second cone is perpendicular to a plane touching the first cone; this plane will necessarily pass through the perpendicular to the plane P, which is the side A of the first cone; it will therefore touch this cone along this side A. Therefore,

To a side of the first cone and to the tangent plane passing through this side, correspond a tangent plane to the second cone and the side of contact with this tangent plane.

8. To two planes touching the first cone correspond two sides of the second cone; the plane of these two sides is perpen-

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dicular to the right line of intersection of the two tangent planes, and the planes touching the second cone along these sides are perpendicular to the sides of contact of the two planes touching the first cone; their right line of intersection is perpendicular to the plane of these two sides of contact; therefore,

To a right line and to its polar plane, with relation to the first cone, correspond a plane and its polar, with relation to the second cone.

9. Hence it follows, that *to two conjugate axes of the first cone correspond two conjugate planes of the second cone.*

For the polars of these two planes, with relation to the second cone, will correspond to the two polar planes of the two axes of the first cone; these two planes mutually pass through these two axes, since these axes are conjugate; therefore, the two right lines which correspond to these planes are mutually in the diametral planes of the second cone, which proves that these two planes are conjugate (1).

10. To two right lines drawn arbitrarily through the vertex of the first cone, will correspond in the second cone two planes, containing between them an angle equal to the supplement of that of the two right lines; the intersection of these two planes will be perpendicular to the plane of the two right lines.

Analogy of supplementary cones, spherical triangles and trihedral angles.

These relations between the two cones are the same as those which exist between two supplementary trihedral angles, or between two supplementary spherical triangles; on this account we shall call the two cones *supplementary* one to the other.

From what has been said, it is plain that the properties relative to the angles contained by certain planes and right lines passing through the vertex of the first cone will give rise to properties of corresponding right lines and planes in the second cone; so that *the properties of cones of the second degree are double*, as well as those of spherical triangles.

11. Let us suppose that the two supplementary cones have the same vertex.

The line joining the centre of the Cone to the vertex is the third diameter.

To three conjugate diameters of the first cone will evidently correspond three conjugate diametral planes of the second cone (9). Therefore, to the three rectangular conjugate diameters of the first cone will correspond the three rectangular conjugate diametral planes of the second. To the principal axis of the first cone will correspond the principal plane of the second; so that the two cones will have the same principal axis and the same principal plane. But the plane of the greatest section of the first will be the plane of the least sec-

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tion of the second, since the angle between the two sides of the second cone contained in this plane will be the supplement of the angle between the two sides of the first cone contained in this plane: in like manner, the plane of the least section of the first cone will be the plane of the greatest section of the second. It follows, that the major axis of the first cone will be the minor axis of the second, and the minor axis of the first cone will be the major axis of the second.

12. It is known that every cone of the second degree may be cut in circular sections by two series of planes parallel to two fixed planes; and that these two fixed planes being drawn through the vertex of the cone, pass through its major axis, and are therefore perpendicular to the plane of the least section.

As these two fixed planes possess a great number of properties which we are about to state, we are under the necessity of designating them by a particular name, and we shall call them *cyclic planes*.

Thus, the *cyclic planes* of a cone of the second degree are two fixed planes passing through its vertex, and parallel to the planes of the circular sections of the cone.

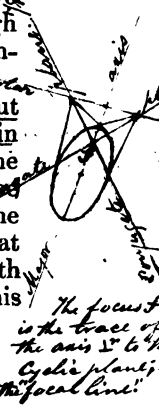
13. Two conjugate axes of the cone contained in a cyclic plane, are always at right angles, since they are parallel to two conjugate diameters of the section of the cone made by a plane parallel to the cyclic plane, this section being a circle; hence,

The cyclic planes of a cone possess this characteristic property, that

Two conjugate axes of the cone contained in a cyclic plane are always at right angles.

14. To the cyclic planes of a cone correspond, in the supplementary cone, two right lines perpendicular to these cyclic planes. Two conjugate axes of the first cone, contained in a cyclic plane, being at right angles (13), we infer that two conjugate planes of the supplementary cone, passing through one of the two right lines just mentioned, are at right angles.

A plane perpendicular to one of these right lines will cut the cone in a conic section, and the two conjugate planes in two right lines perpendicular to each other, and such that the pole of one of them, with relation to the conic section, will be upon the other right line (1). Thus, the point where the cutting plane meets the right line in question is such that every secant passing through this point has its pole, with relation to the conic section, upon the perpendicular to this



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secant passing through the point. This proves that the point *is a focus of the conic section.**

Hence, every plane perpendicular to one of the two right lines just mentioned cuts the cone in a conic section, one of whose foci is upon this right line.

These two right lines, being perpendicular to the two cyclic planes of the first cone, are in the plane of its least section (12), and are consequently in the plane of the greatest section of the supplementary cone (11). We have, therefore, this general property of cones:

In every cone of the second degree there are two right lines lying in the plane of its greatest section, which possess the property that every plane perpendicular to one of them cuts the cone in a conic section, one of whose foci is upon this right line.

In the first memoir above referred to, we proved the existence of these two right lines by means of the theory of polar transformations, and we called them *focal lines* in consequence of the characteristic property which we have just stated; but we did not there observe that *two conjugate planes passing through one of these two right lines are always at right angles.*

15. From what precedes it results, that

In two cones of the second degree, supplementary one to the other, the focal lines of one correspond to the cyclic planes of the other.

This theorem is very important, since it follows from it that the properties of the focal lines are consequences from those of the cyclic planes, and *vice versâ*.

* In what follows, we shall not take for granted any of the properties of the foci of the conic sections; these properties might even be deduced from those which we are about to demonstrate with reference to the cones of the second degree. However, it is necessary to define the foci of a conic section by some one of their properties. That which we have employed here to characterize these points is not, it is true, the most generally known; but it is known, and we are indebted for it to Dé Lahire; M. Poncelet has proved it, and has applied it with advantage in his "Traité des Propriétés Projectives" (p. 260, Nos. 451 and 453.) We shall take another opportunity of showing that this property adapts itself to the study of the conic sections with more readiness than many others equally characteristic. Moreover, it must be remarked, that we here define the foci of a conic section by the property in question merely to justify the designation of *focal lines*, which we adopt for the two axes of a cone of the second degree perpendicular to the cyclic planes of the supplementary cone. But all the properties of these two axes which we are about to prove are quite independent of the foci of the conic sections; and any one of these properties might as well be fixed upon to justify the designation of focal lines. See Salmon Conics p. 168 3rd Ed. 1855 where it is shown that "the angle subtended at the foci by any chord is bisected by the line joining the foci to its pole"; and hence two lines inclined to each other passing through the foci have their poles on each other.

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Thus, it will be sufficient for us to prove the one, and merely to state the others, without actually giving a particular demonstration of these latter.

16. First, let us lay down some properties of the circular sections of a cone, which we shall have to make constant use of.

When we consider two circular sections of a cone whose planes are not parallel, we call them *antiparallel* or *subcontrary* sections.

Through each point on the surface of a cone of the second degree, two subcontrary sections can be made to pass; for it is sufficient to draw through this point two planes parallel to the two cyclic planes of the cone.

17. *Every sphere passing through a circular section of a cone cuts this cone in a second subcontrary circular section.*

For, any side of the cone penetrates the sphere in two points, the rectangle under the distances of which from the vertex of the cone is constant: one of these points is on the plane of the circle through which the sphere has been made to pass; consequently, the other point is upon a sphere passing through the vertex of the cone, and having its centre upon the perpendicular let fall from the vertex upon the plane of this circle; (see the note at the end of this memoir;) the intersection of the first sphere and the cone is upon this second sphere, which proves that this intersection is a circle. The plane of this second circle cannot be parallel to the plane of the first; for in that case the cone would evidently be one of revolution. This second circle is therefore a subcontrary section with relation to the first. The theorem is therefore demonstrated.

It follows from hence, that

A sphere can be made to pass through any two subcontrary sections of a cone of the second degree.

18. Let us suppose that the plane of one of the two circles approaches indefinitely towards the vertex of the cone; as this plane is always parallel to a cyclic plane, when this circle degenerates into a point which is the vertex of the cone, the sphere will touch this cyclic plane; therefore,

Every sphere passing through a circular section and through the vertex of a cone of the second degree, touches a cyclic plane;

And conversely,

Every sphere passing through the vertex of a cone of the second degree, and touching a cyclic plane, cuts the cone in a circle, whose plane is parallel to the second cyclic plane of the cone.

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SECTION II.

PROPERTIES OF THE TWO CYCLIC PLANES OF A CONE CONSIDERED SIMULTANEOUSLY; AND PROPERTIES OF THE TWO FOCAL LINES OF A CONE CONSIDERED SIMULTANEOUSLY.

19. The properties of the cyclic planes of a cone are of two kinds; some have reference to these two planes considered simultaneously, others to only one of these planes. It is as easy to state the one as the other; but as the former appear in some degree to be more characteristic, we shall commence with them.

The two focal lines of a cone, as appears from theorem (15), will be found to possess properties corresponding to those of the two cyclic planes.

20. Let there be a cone of the second degree; let it be cut by two planes parallel to its two cyclic planes; the sections will be two circles lying on the same sphere (17). Each side of the cone meets these circles, and the tangent plane to the cone along this side cuts their planes in two right lines which are the tangents to these circles, passing through the points where the side meets them: these two right lines are therefore tangents to the sphere which passes through the two circles. Now, two tangents to a sphere lying in the same plane make equal angles with the chord which joins the two points of contact; here this chord is the side of the cone; the two tangents are respectively parallel to the two right lines in which the tangent plane intersects the two cyclic planes. We have, therefore, the following property of the cyclic planes, and, consequently, the property of the focal lines stated in the second column:

Every tangent plane to a cone of the second degree intersects the two cyclic planes in two right lines, which make equal angles with the side of the cone along which it is touched by the tangent plane.	The planes passing through the two focal lines of a cone of the second degree and through any side whatever, make equal angles with the plane touching the cone along that side(*a*).

(*a*) This theorem and the second in No. 24, have been already given by M. Magnus, of Berlin (Annales de Mathématiques, August, 1825.) As to all the other properties of the cones of the second degree contained in this memoir, we believe they are quite new, with the exception of some which we have already stated in our Memoir on the surfaces of revolution of the second degree, inserted in the 5th vol. of the New Memoirs of the Royal Academy of Brussels.

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21. Conversely,

If a cone be such that its tangent plane, passing through any side, intersects two fixed planes in two right lines, making equal angles with that side, this cone is of the second degree.

If a cone be such that the planes passing through two fixed axes and through any side whatever, make equal angles with the plane touching the cone along that side, this cone is of the second degree.

In order to demonstrate the first theorem, let us draw two planes P, P′, parallel to the two fixed planes. Let M, M′, M″, M‴, be the points where four consecutive sides infinitely near to one another meet the first plane P, and let m, m', m'', m''', be the points where these same sides meet the second plane P′.

Let us imagine the circle which passes through the three points M, M′, M″, to be the base of a second cone, having the same vertex S, as the proposed cone; the section of this cone made by the second plane P′ will pass through the points m, m', m'', and this curve will be a circle; for (by hypothesis) the two right lines MM′, mm', making equal angles with the side SMm, the plane of the circular section of the second cone passing through the point m will pass through the right line mm', as appears from the preceding theorem; the two tangents M′M″, $m'm''$ also making equal angles with the side SM′m', the plane of this circular section will likewise pass through the tangent $m'm''$; this plane will, therefore, be exactly the plane of the three points m, m', m''; so that P, P′ will be the planes of two subcontrary sections of the second cone: the tangents to these two sections passing through the points M″, m'', will, therefore, make equal angles with the side SM″m'' (20); but the two right lines M″M‴, $m''m'''$ make (by hypothesis) equal angles with that side; these two right lines are, therefore, actually the tangents to the two circular sections of the second cone: for, through the two points M″, m'' only two right lines can be drawn in the planes P, P′, which make equal angles with the side SM″m'', since these two right lines intersect at the point where the plane, drawn at right angles to this side through the middle point of the segment M″m'', meets the intersection of the two planes P, P′.

Hence, it is proved that the circle passing through three points infinitely near, M, M′, M″, assumed in the section of the proposed cone made by a plane parallel to one of the two fixed planes, passes always through a fourth point M‴ of this curve infinitely near the former; for the same reason, this circle, passing through the three consecutive points M′, M″, M‴, will pass through a fifth point, and again, through a sixth; which proves that this base is itself a circle; or, in other words, the osculating circle of this curve has at every point in it a con-

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tact with it of the third order, from which it follows that this curve can only be a circle. Thus we have proved the first theorem, and consequently the second.

22. The two theorems (20) are particular cases of the following:

Every plane passing through two sides of a cone of the second degree intersects the cyclic planes in two right lines which respectively make equal angles with these two sides.

The planes passing through the two focal lines of a cone of the second degree and through the right line of intersection of two tangent planes to the cone respectively make equal angles with these two tangent planes.

It is sufficient to prove the first of these two theorems.

For this purpose, let us take two subcontrary sections of the cone; the plane of the two sides cuts the planes of these two circles along two chords, which form, with the portions of the two sides lying between these chords, a plane quadrilateral, inscribed in the circle in which the plane of this quadrilateral intersects the sphere, on the surface of which are the two subcontrary sections; two opposite angles of this quadrilateral are, therefore, supplemental one to the other; hence the two chords respectively make equal angles with the two sides of the cone. These chords are parallel to the right lines in which the plane of the two sides intersects the two cyclic planes; the first theorem is, therefore, proved; the second follows from it.

In the memoir above referred to we had already proved these two theorems in two different ways; first, as a consequence from the properties of the lines of curvature of a hyperboloid of one sheet; and again, as a consequence from the properties of surfaces of revolution of the second degree.

23. Two planes touching a cone of the second degree along any two sides, intersect the two cyclic planes in four right lines, which are the generatrices of the same cone of revolution, whose axis of revolution is perpendicular to the plane of the two sides of contact.

The four vector planes, passing through the two focal lines of a cone of the second degree and through any two sides of the cone, are tangents to the same cone of revolution, whose axis of revolution is the right line of intersection of the two planes touching the proposed cone along the two sides.

Let us prove the first theorem.

The first tangent plane intersects the two cyclic planes in two right lines, making equal angles with the side of contact (20); therefore, these two right lines make equal angles with every plane passing through this side; consequently, they make equal angles with the plane of the two sides. In like

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manner the second tangent plane intersects the two cyclic planes in two right lines, which make equal angles with this same plane of the two sides.

Now let us imagine a plane parallel to one of the cyclic planes; it will cut the cone in a circle, and the two tangent planes to the cone in two tangents to this circle; these two tangents will be parallel to the two right lines in which the cyclic plane intersects the two tangent planes; but these two tangents make equal angles with the chord which joins their points of contact, and, consequently, they make equal angles with every plane passing through this right line; they, therefore, make equal angles with the plane of the two sides of contact with the tangent planes.

Hence, it follows, that the four right lines of intersection of the two cyclic planes with the two tangent planes, make equal angles with the plane of the two sides; consequently, they make equal angles with the right line perpendicular to this plane, which proves that they are generatrices of the same right cone, whose axis of revolution is perpendicular to the plane of the two sides of contact. Q. E. D.

24. The sum or the difference of the angles, which each tangent plane to a cone of the second degree makes with the two cyclic planes, is constant.

The sum or the difference of the angles, which each side of a cone of the second degree makes with its two focal lines, is constant. (See note to 20.)

These two theorems may be respectively deduced from the two preceding ones in the same manner: we mean to prove the second, since the figure it requires is easily constructed.

Let there be a sphere, whose centre is at the vertex of a cone, it will meet the two focal lines, (supposed to be prolonged within a single sheet of the cone,) in two points F, F′; it will meet any two sides of the cone in two points, m, n; and the four vector planes passing through these sides, in four arcs of great circles, which will touch a small circle of the sphere formed by the intersection of the sphere with the right cone, to which the four vector planes are tangents. (23, second column.)

As the arcs Fm, F′m, Fn, F′n, measure the angles which the two focal lines make with the two sides of the cone, our object is to prove that the sum of the first two is equal to the sum of the two latter.

Let a, a', b, b', be the points where these four arcs respectively touch the small circle.

The two arcs ma, ma' are equal, as being drawn from the same point m, to touch the small circle; hence we conclude that $\text{F}m + \text{F}'m = \text{F}a + \text{F}'a'$.

In like manner, the arcs nb, nb' are equal, and it follows

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that $Fn + F'n = Fb + F'b'$; now, the arcs Fa and Fb are equal, the arcs $F'a'$ and $F'b'$ are also equal; the two right hand members of the two equations are therefore equal, and, consequently, the first members are likewise equal to one another; which was to be proved.

The angles made by the two generatrices of the cone with a focal line, are the supplements of the angles made by these generatrices with the production of this focal line within the second sheet of the cone; so that the difference of the angles made by a side of the cone with one focal line and with the production of the other focal line, is also a constant quantity; it was for this reason that in the enunciation of the theorem we said, *the sum or the difference* of the angles made by each side with the two focal lines. Thus the theorem is proved.

25. It is known, that if, in a spherical triangle, one angle be invariable, and also the product of the trigonometric tangents of the halves of the sides containing it; the area of the triangle will remain constant, (Legendre's Geometry; Note on the Area of the Spherical Triangle.) From the first of the two preceding theorems we may, therefore, deduce the first, and, consequently, the second of the two following ones:

In every cone of the second degree, each tangent plane intersects the two cyclic planes in two right lines such that the product of the tangents of the semi-angles, which they make with the intersection of the two cyclic planes, is constant.

In every cone of the second degree, the vector planes, passing through the two focal lines and through any side of the cone, are such that the product of the tangents of the semi-angles, which they make with the plane of the two focal lines, is constant.

It would be easy to prove these two theorems directly, without making use of the proposition in spherical trigonometry to which we referred; but as this proposition is to be found in an elementary work familiar to all geometers, we may be allowed, in order to save time, to avail ourselves of it; moreover, we shall hereafter give the direct proof of which we speak, and which will exhibit the two theorems above stated, as depending upon the most elementary principles of geometry.

26. Let there be two subcontrary sections of a cone of the second degree; each side of the cone meets these two circles in two points, the rectangle under the distances of which from the vertex of the cone, is invariable, whatever be this side, since these two circles are upon the same sphere (17). If from the vertex of the cone, perpendiculars be let fall on the planes of the two circles, they will be respectively equal to these two distances, respectively multiplied by the sines of the angles, which the side of the cone makes with the planes of

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the two circles; therefore, the rectangle under the two perpendiculars will be equal to the rectangle under the two distances, multiplied by the product of the two sines. Now, the perpendiculars will be the same, whatever side of the cone we consider; the rectangle under the two distances is also the same, as we have just shown; therefore, the product of the two sines is constant; but the planes of the two circles are parallel to the two cyclic planes of the cone; we have, therefore, the first of the two following theorems, and, consequently, the second.

In every cone of the second degree, the product of the sines of the angles, which each side makes with the two cyclic planes, is constant.

In every cone of the second degree, the product of the sines of the angles, which each tangent plane makes with the two focal lines, is constant.

27. If from a point o, taken arbitrarily, perpendiculars be let fall upon the tangent planes to a cone of the second degree, they will form a second cone of the second degree, which will be the supplementary one to the first (10). The feet of these perpendiculars will be upon the sphere whose diameter is the right line joining the point o with the vertex of the given cone, since the right line drawn from this vertex to the foot of each perpendicular, makes a right angle with that perpendicular. Hence the feet of the perpendiculars will be upon the curve of intersection of the sphere and the second cone.

Let us suppose the point o to be upon a focal line of the given cone. This right line is perpendicular to a cyclic plane of the second cone (15); the sphere whose centre is upon this right line will, therefore, touch this cyclic plane, which proves that it will intersect the second cone in a circle lying in a plane parallel to the second cyclic plane of this cone (18); this second cyclic plane is perpendicular to the second focal line of the given cone (15); we have, therefore, the following theorem:

If from a point assumed upon a focal line of a cone of the second degree, perpendiculars be let fall upon the tangent planes to this cone, their feet will be upon a circle, the plane of which will be perpendicular to the second focal line of the cone.

28. The right lines which join the vertex of the given cone with the feet of the perpendiculars are the orthogonal projections of the first focal line upon the tangent planes; these right lines form a cone having for its base the circle which is the locus of the feet of the perpendiculars. This cone is evidently symmetrical on each side of the plane of the greatest section of the given cone in which the focal line lies;

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and it touches this given cone along the two sides lying in this plane, since these two sides are the projections of the focal line upon the tangent planes along those sides. From this position of the cone it results, that the orthogonal projections of the second focal line upon the tangent planes will be upon this same cone. This cone will therefore pass, as appears from the theorem which we have just proved, through a second circle lying in a plane perpendicular to the second focal line; whence we deduce the first of the two following theorems, and consequently the second:

The orthogonal projections of the two focal lines of a cone of the second degree upon the tangent planes to the cone form a second cone of the second degree, which has a double contact with the proposed cone, and whose cyclic planes are perpendicular to the focal lines of the latter.

If a right angle, having the same vertex with a cone of the second degree, turn round this vertex, so that one of its sides moves along either of the two cyclic planes of the cone, whilst its other side moves round the cone; the plane of this angle will envelope a cone of the second degree, which will have a double contact with the given cone, and whose focal lines will be perpendicular to the two cyclic planes of the latter.

29. Theorem (27) gives rise to the following, which is the converse of it:

If through the different points of a circle traced upon a cone of the second degree planes be drawn perpendicular to the sides passing through these points, these planes will envelope a second cone of the second degree, one of whose focal lines will pass through the vertex of the given cone, and whose other focal line will be perpendicular to the plane of the circle.

30. If from a point in the focal line of a cone, perpendiculars be let fall upon two tangent planes, the right line joining their feet will be at right angles with the right line of intersection of the two tangent planes; now, by theorem (27), the former right line will also be at right angles with the second focal line of the cone; it is, therefore, perpendicular to the plane passing through the right line of intersection of the two tangent planes and through the second focal line; therefore,

If from a point in a focal line of a cone of the second degree, perpendiculars be let fall upon two tangent planes, the plane passing through the vertex of the cone and perpendicular to the right line joining their feet will pass through the second focal line of the cone.

31. This may be otherwise stated in the first of the two following theorems, from which the second is a consequence:

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If a focal line of a cone of the second degree be orthogonally projected upon two tangent planes, the plane passing through the right line of intersection of the two tangent planes, and perpendicular to the plane containing the two projections, will pass through the second focal line.

If two right angles have each a side upon a cyclic plane of a cone of the second degree, and their two other sides coincident with two sides of the cone, the right line of intersection of their planes and the right line in which the plane of the two sides intersects the second cyclic plane will be at right angles.

32. Let there be two cones of the second degree, having the same vertex and the same cyclic planes; if a common tangent plane be drawn to them, it will touch the two cones along two sides, each of which will make equal angles with the two right lines in which this tangent plane intersects the two cyclic planes (20.) These two sides will therefore bisect both the angle and the supplement of the angle contained by these two right lines, which proves that these two sides are at right angles. We have, therefore, the two following theorems:

When two cones of the second degree have the same vertex and the same cyclic planes, if a common tangent plane be drawn to them, the two sides of contact lying in this plane will be at right angles to each other.

If two cones of the second degree, which have the same vertex and the same focal lines, intersect one another, their tangent planes passing through each side of intersection will be at right angles, that is to say, the two cones will cut one another at right angles.

M. Dupin and M. Binet, Jun. have stated the general conditions to be fulfilled in order that two surfaces of the second degree may cut every where at right angles; but they did not include in their researches the case of two conical surfaces, which is disposed of in the second of the two theorems just proved. (See Développemens de Géométrie de Ch. Dupin, and, 16^{e} cahier des Journaux de l'école Polytechnique.)

These two theorems may be employed with great advantage in different investigations, as we shall hereafter have occasion to show.

SECTION III.

PROPERTIES OF A CONE OF THE SECOND DEGREE RELATING TO A SINGLE CYCLIC PLANE; AND PROPERTIES RELATING TO A SINGLE FOCAL LINE.

33. The properties of a cone of the second degree relating to a single cyclic plane, which we are about to prove, are deduced immediately from the properties of the circle; and those relative to a single focal line may be shewn to depend

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upon the former by reference to the supplementary cone. Hence, we shall content ourselves with merely writing them alongside of the former.

The right line polar to a plane, with relation to a cone, passes, as we have stated (2,) through the centre of the section made in the cone by this plane or by any other parallel to it. Hence, the polar of a cyclic plane is the right line which is the locus of the centres of the circular sections of the cone lying in planes parallel to this cyclic plane.

We have seen that, to a cyclic plane and to its polar, correspond, in the supplementary cone, a focal line and its polar plane (14.) We shall call this polar plane the *director plane* of the cone, a name analogous to that of directrix in the conic sections. A cone has two director planes, but we must always be understood to refer to that which corresponds to the focal line which we are considering.

34. Let us begin with two theorems, the second of which is remarkable for its analogy to the leading property of the directrices of the conic sections.

In every cone of the second degree the ratio of the sines of the angles made by each tangent plane with a cyclic plane and with the polar of this cyclic plane is constant.

In every cone of the second degree the ratio of the sines of the angles which each side makes with a focal line and with the director plane is constant.

To prove the first of these two theorems, let us take a cone of the second degree and a secant plane parallel to one of its cyclic planes; the section of the cone will be circle, and the axis of the cone passing through the centre of this circle will be the polar of the cyclic plane. Let there be a tangent plane to the cone, and from the centre of the circle let us drop a perpendicular on this plane; the sine of the angle made by this tangent plane with the plane of the circle is equal to this perpendicular divided by the radius of the circle; the sine of the angle made by the tangent plane with the axis of the cone terminating in the centre of the circle is equal to this same perpendicular divided by the length of this axis. From the form of these expressions for the sines of the two angles, it appears that their ratio is independent of the perpendicular, and contains only the radius of the circle and the distance of its centre from the vertex of the cone. Hence, this ratio is constant, whatever be the tangent plane to the cone: which proves the first of the two theorems above stated; the second is deduced from it by reference to the supplementary cone.

35. In what follows we shall still consider, as we have just done, a circular section of the cone, and we shall regard the

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points and right lines lying in the plane of this section as the intersections of this plane, with right lines and planes passing through the vertex of the cone; and, in order to apply to the cone the various properties of the circle, in place of the right lines lying in the plane of this circle, we shall substitute, in the statement of our theorems, the traces upon the cyclic plane of the planes passing through these right lines and through the vertex of the cone.

36. Thus, in order to transfer to the cone, this property of the circle, "a radius is perpendicular to the tangent passing through its extremity," we shall consider the tangent and the radius as being the traces of a tangent plane to the cone and of a plane passing through the side of contact and through the polar of the cyclic plane (since this polar passes through the centre of the circle;) and we shall remark that these two traces are respectively parallel to the traces of the same two planes upon the cyclic plane parallel to the plane of the circle; whence we have the first of the two following theorems, and, consequently, the second.

The tangent plane to a cone of the second degree, and the plane passing through the side of contact and through the polar of a cyclic plane, meet this cyclic plane in two right lines which are at right angles with each other.

If through a focal line of a cone of the second degree two vector planes be drawn, of which the first passes through any side of the cone, and the second through the right line in which the plane touching the cone along that side meets the director plane, these two vector planes will be at right angles.

37. Two tangents to a circle make equal angles with the chord which joins the two points of contact. Hence we infer, in accordance with what has been said (35,) that

Two tangent planes to a cone of the second degree and the plane of the two sides of contact intersect a cyclic plane in three right lines, the third of which bisects the angle between the first two.

If through a focal line of a cone of the second degree vector planes be drawn respectively passing through two sides of the cone and through the right line of intersection of the two planes touching the cone along these sides, the third vector plane will bisect the angle between the first two.

38. Two tangents to a circle make equal angles with the right line which joins their point of concourse with the centre of the circle. Hence we infer, that

Two tangent planes to a cone of the second degree, and the plane passing through their right line of intersection and through the polar

The vector planes passing through a focal line of a cone of the second degree and through two sides make equal angles with the vector plane

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of a cyclic plane, meet that cyclic plane in three right lines, the third of which bisects the angle between the first two.

passing through the right line in which the plane of the two sides intersects the director plane.

39. The right line drawn from the centre of a circle to the point of concourse of two tangents is perpendicular to the chord which joins the two points of contact, and passes through the middle of this chord. Let it be observed, that the middle point of a right line is the harmonic conjugate to a point in this line lying at an infinite distance, and that the four right lines drawn from any point whatsoever to four harmonic points form a harmonic pencil; and we shall have the first of the two following theorems, and, consequently, the second.

The plane passing through the polar of a cyclic plane of a cone of the second degree and through the right line of intersection of two tangent planes to the cone, and the plane of the two sides of contact, meet the cyclic plane in two right lines which are at right angles with each other;

And the right lines, in which the the plane of the two sides meets the cyclic plane and the plane passing through its polar and through the right line of intersection of the two tangent planes, are harmonic conjugates with relation to the two sides.

Two vector planes passing through a focal line of a cone of the second degree will be at right angles, if one passes through the right line of intersection of two tangent planes to the cone, and the other through the right line in which the plane of the two sides of contact meets the director plane;

And the two planes drawn through the right line of intersection of the two tangent planes, and passing, the first through the focal line, and the other through the right line in which the plane of the two sides meets the director plane, are harmonic conjugates with relation to the two tangent planes.

40. *If the plane of the two sides passes through the polar of the cyclic plane, the right line of intersection of the two tangent planes will be in this cyclic plane; therefore,*

If through a right line lying in the cyclic plane of a cone of the second degree two tangent planes be drawn to the cone, the plane of the two sides of contact will intersect the cyclic plane in a second right line which will be at right angles to the former.

If the right line of intersection of the two tangent planes be in the director plane, the plane of the two sides of contact will pass through the focal line; therefore,

If a cone of the second degree be cut by a transversal plane passing through a focal line, the planes touching the cone along the two sides lying in this plane will intersect on the director plane; and the plane passing through the focal line and through the right line of intersection of the two tangent planes will be perpendicular to the transversal plane.

41. When a quadrilateral is inscribed in a circle two opposite angles are always supplemental one to the other. Hence we infer, that

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When a tetrahedral angle is inscribed in a cone of the second degree, the angle between the traces of two of its adjacent faces upon a cyclic plane of the cone is supplemental to the angle contained by the traces upon the same cyclic plane of the two other faces.

When a tetrahedral angle is circumscribed about a cone of the second degree, the vector planes, passing through a focal line and through two adjacent edges of the tetrahedral angle, contain an angle which is supplemental to the angle contained by the vector planes passing through the two other edges.

42. All the angles whose vertices are upon the circumference of a circle, and whose sides pass through the extremities of the same chord, are equal (we mean the acute angle contained between the two sides, otherwise two angles would be supplemental one to the other, when their vertices were on opposite sides of the chord;) if this chord pass through the centre of the circle all the angles are right. Hence we infer, that

If through two fixed sides of a cone of the second degree, two planes be drawn, intersecting along any third side of the cone, the traces of these planes upon a cyclic plane will contain between them an angle of invariable magnitude.

Two fixed tangent planes being drawn to a cone of the second degree, and also any third tangent plane whatever, the vector planes passing through a focal line and through the two right lines in which the two fixed planes are intersected by the third tangent plane will contain between them an angle of invariable magnitude, whatever be this third tangent plane.

This angle will be right if the plane of the two fixed sides passes through the polar of the cyclic plane.

This angle will be right if the right line of intersection of the two fixed tangent planes be in the director plane corresponding to the focal line.

43. *The polar of a cyclic plane is in the plane of the least section of the cone. Hence we infer from the second part of the preceding theorem, that*

The director plane passes through the minor axis of the cone; the two planes touching the cone along the sides lying in the plane of the greatest section also pass through the minor axis; they therefore intersect upon the director plane. Hence we infer from the second part of the preceding theorem, that

If through the two sides of a cone of the second degree lying in the plane of its least section two planes be drawn intersecting along any side whatever of the cone, their traces upon one of the cyclic planes will be at right angles.

Every tangent plane to a cone of the second degree intersects the two tangent planes that are perpendicular to the plane of the greatest section in two right lines such that the vector planes, passing through a focal line and through these two right lines, are at right angles to each other.

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44. From the two theorems (42,) the following immediately result:

If in a cyclic plane of a cone of the second degree, an angle of invariable magnitude, and having the same vertex with the cone, be made to turn; the plane passing through one side of the angle, and through a fixed side of the cone, will pass through a second side; and the plane determined by this second side of the cone and by the second side of the moveable angle will turn round a fixed side of the cone.

If round a focal line of a cone of the second degree, as an edge, a dihedral angle of invariable magnitude be made to turn; and if through the right line in which one of its faces meets a fixed tangent plane to the cone a second tangent plane to the cone be drawn, this second plane will meet the second face of the dihedral angle in a right line which will always be in the same tangent plane to the cone.

45. If, from the vertex of a cone, a perpendicular be let fall upon the plane of a circular section of the cone, and two parallel tangents be drawn to this circle, the sum of the distances of the foot of the perpendicular from the two tangents, will be constant and equal to the diameter of the circle; but, these distances may be taken as the trigonometric tangents of the angles, which the two tangent planes to the cone, passing through the two tangents to the circle, make with the perdicular let fall from the vertex of the cone, and these trigonometric tangents are equal to the reciprocals of the trigonometric tangents of the angles which the two tangent planes make with the plane of the circle, or with the cyclic plane to which the plane of the circle is parallel; hence we infer, that

If, through a right line lying in a cyclic plane of a cone of the second degree, two tangent planes be drawn to the cone, the sum of the reciprocals of the trigonometric tangents of the angles which they make with the cyclic plane will be constant.

Every plane passing through a focal line of a cone of the second degree cuts the cone along two sides such that the sum of the reciprocals of the trigonometric tangents of the angles which they make with this focal line is constant.

SECTION IV.

GEOMETRIC LOCI RELATING TO THE CYCLIC PLANES, AND TO THE FOCAL LINES OF CONES OF THE SECOND DEGREE.

46. If the sides of an angle of invariable magnitude, contained in a given plane, turn round two fixed points, its vertex generates an arc of a circle; hence we infer, that

If round two fixed right lines which intersect, two planes be made

Being given two fixed planes and a right line passing through a point

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to turn, so that their traces upon a fixed plane passing through the point of intersection of the two right lines may contain an angle of invariable magnitude, the intersection of the two moveable planes will generate a cone of the second degree, which will have the fixed plane for one of its cyclic planes, and which will pass through the two fixed right lines.

in their intersection, if round this right line two planes be made to turn, containing between them an angle of invariable magnitude, the two right lines in which these planes will respectively intersect the two fixed planes will determine a moveable plane, which will envelope a cone of the second degree, in which the fixed right line will be a focal line, and which will touch the two fixed planes.

It is scarcely necessary to mention, that, as upon a given right line, two arcs of circles can be described capable of containing the same angle; so, in each of these two propositions, there are two cones perfectly equal.

We might have deduced these two propositions directly from the two theorems (42.)

47. The vertex of an angle of invariable magnitude, whose sides touch a circle, generates a second circle; and the chord which joins the point of contact of the two sides, envelopes a third circle; these three circles are concentric; whence we infer, that

If two tangent planes to a cone of the second degree move in such a way that their traces upon a cyclic plane contain between them an angle of invariable magnitude, the intersection of these two planes will generate a second cone of the second degree.

The plane of the two sides of contact of the two tangent planes will envelope a third cone of the second degree.

The cyclic plane in question will be a cyclic plane of the two new cones, and this plane will have the same polar in the three cones.

If a dihedral angle of invariable magnitude turn round a focal line of a cone of the second degree, as an edge, the plane of the two sides along which its faces will meet the cone will envelope a second cone of the second degree.

The planes touching the given cone along these two sides will intersect upon a third cone of the second degree.

The focal line, round which the dihedral angle turns, will be a focal line of the two new cones; and the corresponding director plane will be the same in the three cones.

48. If an angle of invariable magnitude turn round its vertex, which lies upon the circumference of a circle, the chord intercepted between its sides will always be a tangent to another circle concentric with the former; therefore,

If round a side of a cone of the second degree two planes be made to turn, whose traces upon a cyclic plane contain between them an angle of invariable magnitude, the

If round a focal line of a cone of the second degree a dihedral angle of invariable magnitude be made to turn, and if through the right lines in which the faces of this angle meet

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plane of the two sides along which these two planes meet the cone will envelope a second cone of the second degree, which will have the same cyclic plane with the given one; and this plane will have the same polar in the two cones.

a fixed plane touching the cone two other tangent planes to the cone be drawn, their right line of intersection will generate a second cone of the second degree, in which the edge of the dihedral angle will be a focal line, and the corresponding director plane will be the same in the two cones.

49. Let us take a cone of the second degree, and cut it by any plane whatsoever, the curve of intersection will be a conic section; and it is known, that the vertices of all the right angles, circumscribed about this curve, are upon a circle which has the same centre with it; whence we infer, that

If two tangent planes be drawn to a cone of the second degree, so that their traces upon a fixed plane may contain between them a right angle, the intersection of these two planes will generate a second cone of the second degree, one of whose cyclic planes will be parallel to the fixed plane.

This plane will have the same polar in the two cones.

If two rectangular planes turn round a fixed right line passing through the vertex of a cone of the second degree, the sides along which these two planes cut the cone, taken two by two, will determine four planes which will envelope a second cone of the second degree, in which the fixed right line will be a focal line.

This right line will have the same polar plane with relation to the two cones.

50. If the fixed plane, in the former of these two theorems, be one of the three rectangular conjugate planes of the given cone, the second cone will evidently be one of revolution round the axis perpendicular to this plane; therefore,

If two tangent planes be made to revolve round a cone of the second degree, so that their traces upon one of the three rectangular conjugate planes of the cone shall always be at right angles, the intersection of these two planes will generate a cone of revolution round the axis perpendicular to that plane.

If round one of the three rectangular conjugate axes of a cone of the second degree two rectangular planes be made to turn, the sides along which these two planes will intersect the cone, taken two by two, will determine four planes which will envelope a cone of revolution round that axis.

51. It is known, that if the vertex of an angle of invariable magnitude traverse a right line, whilst one of its sides turns round a fixed point, its other side envelopes a parabola touching the right line traversed by the vertex of the angle. This proposition will help us in the proof of the following theorems:

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If the faces of a variable dihedral angle inscribed in a cone of the second degree turn round two fixed sides of the cone, the plane determined by the two right lines in which these faces intersect the two cyclic planes of the cone will envelope a second cone of the second degree touching these two cyclic planes.

Two fixed planes being drawn so as to touch a cone of the second degree, a third moveable tangent plane will intersect them in two right lines, and the planes respectively passing through these two right lines, and through the two focal lines of the cone, will intersect in a right line, which will generate a cone of the second degree passing through these two focal lines.

It is sufficient to prove the first of these two theorems, since the second may be deduced from it by reference to the supplementary cone.

Let us draw a transversal plane parallel to a cyclic plane of the given cone; this plane will intersect the cone in a circle, the second cyclic plane in a right line D, and the faces of the dihedral angle in two right lines, L, L′, which will pass through two fixed points in the circle, and will intersect in any third point of this circle; the plane, of which we seek the enveloping surface, will be cut by the transversal plane in a right line which will pass through the two points in which the two right lines, L, L′, respectively meet the traces of the two cyclic planes upon the transversal plane; one of these traces is the right line D, the other is at an infinite distance. If, therefore, through the point in which the right line L meets the right line D, we draw a right line parallel to the right line L′, this parallel will be the trace of the plane whose enveloping surface we are seeking. Now, this trace envelopes a parabola; for it makes with the right line L an angle which is of invariable magnitude, as being equal to the angle between the two right lines, L, L′; the vertex of this angle moves along the fixed right line D; its side L turns round a fixed point in the circle; therefore, its second side envelopes a parabola touching the right line D; which proves that the moveable plane envelopes a cone of the second degree touching the cyclic plane, which intersects the transversal plane in the right line D. In like manner it may be proved that this cone will touch the second cyclic plane; the theorem is therefore proved.

52. If, round two fixed right lines which meet another, two rectangular planes be made to turn, their intersection will generate a cone of the second degree, which will pass through the two fixed right lines, and whose cyclic planes will be perpendicular to these two right lines.

If, round a point assumed in the intersection of two fixed planes, a right angle be made to turn, whose sides move in the two fixed planes, the plane of this angle will envelope a cone of the second degree, which will touch the two fixed planes, and whose focal lines will be perpendicular to these two planes.

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For, when two planes are rectangular, every plane perpendicular to one of them intersects them in two right lines which are at right angles to each other; therefore, a transversal plane perpendicular to one of the two fixed right lines intersects the two moveable planes in two right lines which are at right angles to each other; now, these two right lines pass through the two fixed points in which the transversal plane meets the two fixed right lines; their point of intersection will, therefore, generate a circle which passes through these two points, and along which moves the right line of intersection of the two moveable planes; which proves the theorem.

For, let us draw through the vertex of the moveable angle a right line perpendicular to one of the two fixed planes; it will be perpendicular to the side of the angle which moves in this plane; the plane determined by this right line and by the other side, will, therefore, be perpendicular to the former side; whence it follows, that the planes passing through the right line and through the two sides of the angle will be rectangular. Now, we have seen that, when two rectangular planes turn round a fixed right line, the two right lines, in which they intersect two fixed planes passing through a point in this right line, are in a plane which envelopes a cone of the second degree, in which the fixed right line is a focal line (46, *second column*;) *which proves the theorem.*

It would have been sufficient to prove one of these two theorems, since either might be deduced from the other by reference to the supplementary cone.

53. If, round a fixed point, a right line be made to turn, which makes, with two fixed planes, angles, the product of whose sines is constant, this right line will generate a cone of the second degree whose two cyclic planes will be parallel to the two given planes.

If, round a fixed point, a plane be made to turn, which makes, with two given right lines, angles, the product of whose sines is constant, this plane will envelope a cone of the second degree whose focal lines will be parallel to the two given right lines.

These two propositions are evidently the converses of the two theorems (26.)

It is sufficient to prove the first.

For this purpose, let us take O as the fixed point; M, m, as the points in which a side of the cone generated by the moving right line meets the two fixed planes; and P, p, as the feet of the perpendiculars let fall from the point O upon these planes; the sines of the angles which this side makes with these planes are respectively equal to $\frac{OP}{OM}$, $\frac{Op}{Om}$; the product of these two ratios ought therefore to be constant. Now, the numerators are constant, whatever side of the cone we consider; therefore, the rectangle under the two lines, OM, Om, is constant; which proves that the point M being upon a plane, the point m is necessarily upon a sphere, (Note at the end of the Memoir;) this point m lies, therefore, on the intersection of the second given plane and this sphere. Hence, the cone generated by the moving

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right line is cut in circular sections by the two fixed planes; which proves the theorem.

54. A plane and a right line being given, if round their point of intersection we conceive a plane to turn which makes, with the given plane and right line, angles, the ratio of whose sines is constant, this plane will envelope a cone of the second degree, which will have the given plane for one of its cyclic planes; and the given right line will be the polar of this cyclic plane with relation to the cone.

A right line and a plane being given, if round their point of intersection we conceive a right line to turn which makes, with the given right line and plane, angles, the ratio of whose sines is constant, this right line will generate a cone of the second degree, in which the given right line will be a focal line, and the given plane will be the director plane corresponding to that focal line.

These two propositions are the converses of the two theorems (34.)

It is sufficient to prove the first.

For this purpose, let us draw a transversal plane parallel to the given plane; it will meet the given right line in a point A, and the moveable plane in a right line D.

The sine of the angle which this moveable plane makes with the fixed right line is equal to the perpendicular let fall from the point A upon the moveable plane divided by AO (o being the point of intersection of the given plane and right line, round which the moveable plane turns;) the sine of the angle which this moveable plane makes with the transversal plane is equal to the same perpendicular divided by the distance of the point A from the right line D. The ratio of the two sines is therefore equal to this distance of the point A from the right line D, divided by the distance OA; this ratio is constant by hypothesis; the distance OA is likewise constant; consequently, the distance of the point A from the right line D is constant; thus, the trace of the moveable plane upon the transversal plane is always at the same distance from the point A; which proves that this trace envelopes a circle whose centre is at the point A; the cone enveloping the moveable plane is therefore one of the second degree, having a cyclic plane parallel to the transversal plane, and the right line OA is the polar of this cyclic plane; which proves the theorem.

55. In the second of the two preceding theorems it is to be observed, that if from a point in the moveable right line perpendiculars be let fall on the given right line and plane, the ratio of these perpendiculars will be the same as the ratio of the sines of the angles which the moveable right line makes with the given right line and plane. We have, therefore, the following theorem:

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The geometric locus of a point whose distances from a fixed right line and plane are in a given ratio, is a cone of the second degree, which has the given right line and plane for a focal line and its corresponding director plane.

If the ratio of the distances be one of equality, we infer that

The surface, in which every point is equidistant from a given right line and plane, is a cone of the second degree, having the given right line and plane for a focal line and its corresponding director plane.

M. Hachette has employed this theorem in the descriptive solution of the question: "to find the centre of a sphere touching a plane and circumscribed about a hyperboloid of revolution." (See M. Quetelet's Correspondance Mathématique et Physique, Vol. IV. p. 285.) As M. Hachette's proof is extremely simple, we subjoin it.

Let a transversal plane be drawn parallel to the given plane, and let a right circular cylinder be described, having for its axis the given right line, and for its radius the distance of the transversal plane from the given plane. The transversal plane intersects the cylinder in an ellipse, every point of which is equidistant from the given right line and plane. Hence we perceive, that on the cone whose base is this ellipse, and whose vertex is the point of concourse of the given right line and plane, every point is equidistant from this right line and plane.

SECTION V.

PROBLEMS RELATING TO THE CYCLIC PLANES AND TO THE FOCAL LINES OF CONES OF THE SECOND DEGREE; AND GENERAL PROPERTIES OF TRIHEDRAL AND TETRAHEDRAL ANGLES.

56. When the vertex and a cyclic plane of a cone of the second degree are given, only three other conditions are required to determine this cone.

For, if a transversal plane be drawn parallel to the given cyclic plane, this plane will intersect the cone in a circle, which it will be necessary to determine. For this purpose we require three conditions. Hence, the modes of describing a circle subject to three given conditions, will become applicable to the analogous questions relating to a cone of the second degree, of which we have the vertex and one cyclic plane given.

By reference to the supplementary cone, it appears that the vertex and a focal line of a cone of the second degree being given, we require three other conditions to determine this

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cone; and the modes of construction, will correspond to those relating to the analogous questions on cones of the second degree, of which the vertex and a cyclic plane are given.

57. The centre of a circle which passes through two given points is upon the perpendicular erected at the middle point of the right line connecting the two points. Hence, in accordance with what has been already said (39) we deduce the two following theorems:

If a cyclic plane and two sides of a cone of the second degree be given, the polar of this cyclic plane is upon a plane determined by the two following conditions:

1. Its trace upon the plane of the two sides must be the harmonic conjugate, with relation to the two sides, of the right line in which the plane of these two sides intersects the cyclic plane.

2. Its trace upon the cyclic plane must be perpendicular to this right line of intersection.

If a focal line and two tangent planes of a cone of the second degree be given, the director plane corresponding to this focal line, will pass through a right line which is the intersection of the two following planes:

1. The plane which passes through the intersection of the two given tangent planes, and is the harmonic conjugate, with relation to these two planes, of the plane passing through this intersection and through the focal line.

2. The plane passing through this focal line at right angles with this latter plane.

58. The centre of a circle touching two right lines is upon the right line which bisects the angle or the supplement of the angle between them. Hence we infer, that

If a cyclic plane and two tangent planes of a cone of the second degree be given, the polar of this cyclic plane lies in the plane which passes through the intersection of the two tangent planes, and whose trace upon the cyclic plane bisects the angle or the supplement of the angle between the traces of the two tangent planes upon this cyclic plane.

If a focal line and two sides of a cone of the second degree be given, the director plane corresponding to this focal line passes through the right line in which the plane of the two sides meets the vector plane bisecting the angle or the supplement of the angle between the two vector planes passing through the focal line and through the two sides.

59. *Problem.*—Given three sides and a cyclic plane of a cone of the second degree, to determine the polar of this cyclic plane.

Problem.—Given three tangent planes and a focal line of a cone of the second degree, to determine the director plane corresponding to this focal line.

The solutions of these two problems are evidently furnished by the two theorems (57;) it is, therefore, unnecessary to give them.

Problem.—Given three tangent planes and a cyclic plane of a cone of the second degree, to determine the polar of that cyclic plane.

Problem.—Given three sides and a focal line of a cone of the second degree, to determine the director plane of the cone corresponding to that focal line.

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The solutions of these two problems are evidently furnished by the two theorems (58 ;) it is, therefore, unnecessary for us to give them. But it ought to be remarked, that just as the problem of describing a circle touching three given right lines has four solutions, so each of these two problems will likewise admit of four solutions.

61. From what has been said it is plain, that the solution of this question, "to describe a circle touching three circles lying in the same plane," furnishes immediately the solutions of the two following problems:

Problem.—Given three cones of the second degree, having the same vertex and a common cyclic plane, to describe a fourth cone of the second degree, touching these three cones, and having the same cyclic plane with them.

Problem.—Given three cones of the second degree, having the same vertex and the same focal line, to determine a fourth cone of the second degree touching these three cones, and having the same focal line with them.

Each of these problems, generally speaking, admits of eight solutions. The given cones may become planes or right lines, in the same way as we have shewn, (Annales de Mathématiques de M. Gergonne,) that, in the construction of a circle touching three given circles, these latter may become right lines or points.

62. *We have seen* (59,) *that when we have a trihedral angle and a plane passing through its vertex, this plane may be considered as the cyclic plane of a cone of the second degree, three of whose generatrices are the three edges of the trihedral angle; consequently, we infer from theorem* 22, *that*

If, being given a trihedral angle and a transversal plane passing through its vertex, we draw in each face of this angle a right line through the vertex, which makes, with one of the two edges lying in this face, an angle equal to that contained between the other edge and the trace of the transversal plane upon this face, the three right lines thus drawn in the three faces are in the same plane.

This plane and the given transversal plane are the cyclic planes of a cone of the second degree, circumscribed about the trihedral angle.

We have seen (59,) *that when we have a trihedral angle and a right line passing through its vertex, this right line may be considered as the focal line of a cone of the second degree touching the three faces of the trihedral angle; consequently, we infer from theorem* 22, *that*

If, being given a trihedral angle and a right line passing through its vertex, we draw through each edge of this trihedral angle a plane, making, with one of the two faces adjacent to this edge, an angle equal to that which the other face makes with the plane determined by the same edge and by the given right line, the three planes thus drawn pass through the same right line.

This right line and the given right line are the focal lines of a cone of the second degree inscribed in the trihedral angle.

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63. *This theorem leads to the solution of the following problem:*

Problem.—Given three sides and cyclic plane of a cone of the second degree, to determine the second cyclic plane of this cone.

This theorem leads to the solution of the following problem:

Problem.—Given three tangent planes and a focal line of a cone of the second degree, to determine the second focal line of this cone.

64. Theorems (28) give rise to the two following:

A right line being drawn through the vertex of a trihedral angle, if it be projected orthogonally upon the three faces of the angle, and through the three projections a cone be made to pass, one of whose cyclic planes is perpendicular to the given right line, this cone will intersect the three faces of the trihedral angle in three new right lines, and the planes passing through these right lines and respectively perpendicular to the three faces will intersect in the same right line, which will be perpendicular to the second cyclic plane of the cone.

A plane being drawn through the vertex of a trihedral angle, if in this plane we draw three right lines passing through the vertex, and respectively perpendicular to the three edges of the angle, each edge and its perpendicular will determine a plane; if a cone of the second degree be described touching the three planes thus determined, and having for a focal line the perpendicular to the given plane, through the edges of the trihedral angle three new tangent planes to the cone may be drawn, and the right lines drawn in these planes respectively perpendicular to the three edges, will all three be in a new plane perpendicular to the second focal line of the cone.

65. Theorem (30) leads to the following:

If from any point perpendiculars be let fall upon the three faces of a trihedral angle, and through each edge of the angle a plane be drawn perpendicular to the right line joining the feet of the perpendiculars upon the two faces adjacent to this edge, the three planes thus drawn will intersect in the same right line.

66. Theorems (31) lead to the following:

A right line being drawn through the vertex of a trihedral angle, its orthogonal projections on the faces of this angle will be the edges of a second trihedral angle; the planes, passing through the edges of the first angle, and respectively perpendicular to the faces of the second, will pass through the same right line.

A plane being drawn through the vertex of a trihedral angle, three right lines drawn in this plane perpendicular to the three edges of the angle will respectively determine with these edges three planes forming a second trihedral angle; the three right lines drawn in the faces of the first angle, and respectively perpendicular to the three edges of the second, will all three be in the same plane.

67. It is known that the circle circumscribed about a triangle, whose three sides touch a parabola, passes through the

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focus of that curve. (Traité des Propriétés Projectives de M. Poncelet, p. 268.) Hence we infer the first, and consequently, the second, of the two following theorems:

If about a cone of the second degree several trihedral angles be circumscribed, and as many cones of the second degree be respectively circumscribed about these trihedral angles, having all of them a plane touching the given cone for their common cyclic plane, all these cones will pass through a common generatrix.

If in a cone of the second degree several trihedral angles be inscribed, and as many cones of the second degree be respectively inscribed in these trihedral angles, having all of them a side of the given cone for their common focal line, all these cones will touch the same plane.

68. *From what precedes we infer by means of theorem* 28, *second column, that*

A cone of the second degree being given, if any trihedral angle be circumscribed about it, and in a fixed plane touching the cone three right lines be taken making right angles with the three edges of the trihedral angle, and if we further conceive a cone touching the planes of these three angles and having for its focal line the perpendicular to the fixed tangent plane, this new cone will touch a fixed plane, whatever be the trihedral angle circumscribed about the given cone.

From what precedes we infer by means of theorem 28, *first column, that*

A cone of the second degree being given, if any trihedral angle be inscribed in it, and through a fixed side of the cone three planes be drawn respectively perpendicular to the faces of this angle, and if we further conceive a cone of the second degree to be described passing through their three right lines of intersection with these faces, and having one of its cyclic planes perpendicular to the fixed side of the given cone, this new cone will pass through a fixed right line, whatever be the trihedral angle inscribed in the given cone.

69. These two theorems lead to the two following properties of tetrahedral angles:

If a tetrahedral angle be given, and also, a fixed plane passing through its vertex, the planes of the four faces of the tetrahedral angle, taken three by three, will form four trihedral angles: now, if in the given plane right lines be taken making right angles with the edges of these trihedral angles, and a cone of the second degree be described, touching the planes of the three right angles corresponding to each trihedral angle, and having one of its focal lines perpendicular to the given plane, the four cones thus determined will all touch the same plane.

If a tetrahedral angle be given, and also, a fixed right line passing through its vertex, the four edges of this angle, taken three by three, will determine four trihedral angles: now, if through the given right line planes be drawn perpendicular to the faces of each of these trihedral angles, and through the right lines of intersection of these planes with these faces a cone of the second degree be described, having one of its cyclic planes perpendicular to the given right line, the four cones thus determined will pass through a common generatrix.

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SECTION VI.

ORGANIC DESCRIPTION OF CONES OF THE SECOND DEGREE.

70. The organic description of the conic sections given by Newton, is founded upon the following theorem:

If any two constant angles turn round two fixed points as vertices, so that two of their sides intersect upon a given right line, the point of intersection of their two other sides will generate a conic section, which will pass through the two fixed points. (See Universal Arithmetic, Vol. i.; and Principia Mathematica, Lib. i, Lemma 21.)

This construction of the conic sections by points, is general.

The cones of the second degree admit of a similar construction, by which their sides are determined; and also of another analogous construction by which their tangent planes may be determined.

These two modes of describing the cones of the second degree are included in the two following theorems:

71. If any two dihedral angles of invariable magnitude whose edges are fixed and meet one another, turn round these edges so that two of their faces intersect on a fixed plane passing through the point of intersection of the two edges, the intersection of their two other faces will generate a cone of the second degree which will pass through the two fixed edges.

For let us conceive a cone of the second degree, whose focal lines are the edges of the two moveable dihedral angles, and which touches the gived fixed plane.

The two first faces of the two dihedral angles intersect upon this plane, by hypothesis; through their right line of intersection, let us draw a plane M, *touching the cone; this plane meets the second face of the first angle in a right line which generates a plane* P, *touching the cone.* (44, *second column.*)

In like manner, the plane M *meets the second face of the second angle in a right line which generates another plane* P', *touching the cone.*

If any two plane angles of invariable magnitude have for their common vertex a fixed point round which they turn in two given planes, in such a manner that the plane determined by two of their sides turns round a fixed right line passing through their common vertex, the plane determined by their two other sides will envelope a cone of the second degree, which will touch the two planes in which the two angles respectively move.

For let us conceive a cone of the second degree, whose cyclic planes are the two planes in which the two angles move, and which passes through the fixed right line.

The plane which contains the two first sides of the two angles turns round this right line, and cuts the cone along another side M; *the plane determined by this side and by the second side of the first angle turns round fixed side of the cone* P. (44, *first column.*)

In like manner, the plane determined by the side M *and by the second side of the second angle turns round another fixed side of the cone* P'.

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Hence the two second faces of the two dihedral angles meet any plane M *touching the cone in two right lines contained in two fixed planes* P, P', *touching this cone; but these two faces respectively pass through the two edges of the dihedral angles which are the focal lines of the cone; therefore, their right line of intersection generates a cone of the second degree, passing through these two fixed edges.* (51, *second column.*) Q. E. D.

Hence the two second sides of the two angles are respectively in two planes which turn round two fixed sides of the cone P, P', *and which intersect upon another indefinite side* M; *but these two sides are respectively in the two cyclic planes of the cone; therefore, their plane will envelope a cone of the second degree, touching these two cyclic planes.* (51, *first column.*) Q. E. D.

It would have been sufficient to demonstrate one of these two theorems, since the other might have been deduced from it by reference to the supplementary cone.

The first part of our proof of the first theorem is analogous to the method which M. Poncelet has employed in proving the theorem of Newton. (Traité des Propriétés Projectives, p. 274.) It is satisfactory to make this comparison, since it furnishes an example of the advantages attending the modes of proof adopted in that learned work, when applied to questions in the geometry of three dimensions as well as to those of plane geometry.

72. *Problem.*—Given five sides of a cone of the second degree, to determine all the other sides of the cone by the movement of two dihedral angles round their edges.

Let A, B, C, D, E, *be the five given sides; let us take the first two* A, B, *as the edges of the two moveable dihedral angles, and conceive that when two faces of these angles simultaneously coincide with the plane of the two sides* A, B, *their two other faces intersect along the third side* C; *so that these two angles are perfectly determined.*

Now let them be turned round their edges A, B, *so that their two faces, which before passed through the side* C, *may intersect along the fourth side* D, *and again along the fifth side* E; *their two first faces which originally coincided with the plane of the two edges* A, B, *will successively intersect in two right lines* D', E'.

Let the right line of intersection

Problem.—Given five tangent planes to a cone of the second degree, to determine all the other tangent planes by the movement of two angles in their respective planes.

Let A, B, C, D, E, *be the five given planes; let us take the first two* A, B, *as the planes of the two moveable angles; the third plane* C, *will intersect them in two right lines which will make two angles with the right line of intersection with these two planes; these will be the two moveable angles.*

Now let them be turned round their common vertex in their respective planes A, B, *so that their two sides, which before were contained in the plane* C, *may lie in the plane* D, *and again in the plane* E; *their two first sides which originally coincided with the right line of intersection of the two planes* A, B, *will successively determine two planes* D', E'.

Let the plane of the same two

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of the same two faces traverse the plane determined by the two right lines D', E', *the right line of intersection of the two other faces will generate the required cone ; as appears from the preceding theorem.*

sides turn round the right line of intersection of these two planes D', E', *the plane of the two other sides of the two moveable angles will assume all the positions of the planes touching the required cone ; as appears from the preceding theorem.*

SECTION VII.

PROPERTIES OF HYPERBOLOIDS DEDUCED FROM THOSE OF CONES OF THE SECOND DEGREE.

73. It is known that a hyperboloid, whether of one or of two sheets, and its asymptotic cone have the same systems of conjugate diameters. Now, we have proved that in every cone of the second degree there are two axes such that two conjugate planes passing through either of them, are always at right angles (14) ; it follows, therefore, that

In every hyperboloid, whether of one or of two sheets, there are two diameters such that two conjugate planes passing through either of them are always at right angles (*a*).

These two diameters are the focal lines of the asymptotic cone of the hyperboloid.

74. These right lines lie within the cone ; consequently they meet the hyperboloid if it be one of two sheets, and do not meet it if it be of one sheet. About a hyperboloid of one sheet, two cylinders might therefore be circumscribed, having their generatrices parallel to the two right lines in question.

Two planes passing through either of these right lines, and mutually conjugate with relation to the hyperboloid, will also be conjugate with relation to the circumscribed cylinder, whose generatrices are parallel to this right line. A transversal plane perpendicular to this right line will cut the cylinder in a conic section, and will intersect the two conjugate planes along two conjugate diameters of that curve ; now these two diameters will be at right angles ; the conic section will therefore be a circle ; whence we infer, that

About every hyperboloid of one sheet two cylinders may be circumscribed, whose bases upon planes perpendicular to their generatrices are circles (*b*).

(*a*) There are two similar right lines in the ellipsoid ; as we shall show hereafter, when proving a more general property of the surfaces of the second degree.

(*b*) This equally applies to the ellipsoid.

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The axes of these cylinders are the focal lines of the asymptotic cone of the hyperboloid.

75. These cylinders possess two characteristic properties which are easily deduced from two properties of cones of the second degree. For it is well known that the sides of the asymptotic cone of a hyperboloid of one sheet are parallel to the generatrices of the hyperboloid; we infer, therefore, from theorem (34,) second column, in consequence of what has been just stated, that,

In every hyperboloid of one sheet, the ratio of the sines of the angles made by each generatrix with the axis of either of the two circumscribed right circular cylinders, and with the diametral plane conjugate to that axis, is constant.

76. Theorem (24,) second column, leads to the following:

In every hyperboloid of one sheet, the sum or the difference of the angles made by each generatrix with the two axes of the circumscribed right circular cylinders, is constant.

77. The properties of the cones of the second degree lead also to two theorems relative to the subcontrary sections of the hyperboloid of one sheet.

For every transversal plane intersects a hyperboloid, and its asymptotic cone in two similar, and similarly placed conic sections. This may be easily shown, for where two surfaces of the second degree touch one another along a plane curve, every plane intersects them in two conic sections, which have a double contact on the right line in which this plane meets the plane of the curve of contact of the two surfaces. If the plane of this curve be at an infinite distance, the two conic sections will have a double contact on a right line lying at an infinite distance, which indicates that the two conic sections are similar, similarly placed, and concentric.

Hence, it follows, that the planes of the circular sections of a hyperboloid are parallel to the cyclic planes of its asymptotic cone.

78. Consequently, theorem (26,) first column, leads to the following:

In every hyperboloid of one sheet, the product of the sines of the angles made by each generatrix with the planes of the subcontrary sections, is constant.

79. Every plane touching a hyperboloid of one sheet passes through two generatrices, and the plane passing through the centre of the hyperboloid intersects the asymptotic cone along two sides parallel to these two generatrices; consequently, theorem (22,) first column, leads to the following:

In every hyperboloid of one sheet, the tangents drawn at any point on its surface to the two circular sections passing

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through this point respectively make equal angles with the two generatrices which meet in this point.

Note referred to in Nos. 17 *and* 53.

Theorem.—*If from a fixed point radii be drawn, terminating at different points of a given plane, and points be assumed on these radii, whose distances from the fixed point are reciprocally proportional to the radii, these points will all be on a sphere which will pass through the fixed point, and whose centre will be on the perpendicular let fall from this point upon the plane.*

For, let O be the fixed point, P the foot of the perpendicular let fall from this point upon the given plane, and M any point whatever in the plane; let us assume upon the radii OP, OM, two points p, m, such that Op, Om, may be reciprocally proportional to OP, OM, that is to say, such that we shall have OP. $Op = a^2$, OM. $Om = a^2$; (a being a constant right line;) the triangles OPM Opm, are plainly similar, the angle m is, therefore, a right angle; which proves that the point m is upon the sphere described upon Op as diameter. Q. E. D.

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MEMOIR

ON

THE GENERAL PROPERTIES

OF THE

SPHERICAL CONICS.

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ON

THE GENERAL PROPERTIES

OF

THE SPHERICAL CONICS.

SECTION I.

PRELIMINARY CONSIDERATIONS.

1. Let there be a cone of the second degree having its vertex at the centre of a sphere. The curve of intersection of these two surfaces is a *line of curvature* of the cone, since it is every where normal to the generatrices of the cone, which are also lines of curvature.

By the intersection of the sphere with the two sheets of the cone, we obtain two curves, which return into themselves, and which are evidently symmetrical with relation to the three principal planes of the cone, in the same manner as the cone itself is symmetrical with relation to these three planes. These two curves are also symmetrical with relation to the three great circles in which these planes intersect the sphere.

Each of these two curves is of double curvature, unless the cone be one of revolution, in which case they are both circles of the sphere. These two curves, in the general case of any cone of the second degree, have the form of an ellipse, and differ less from this curve as the radius of the sphere becomes greater. On this account we may be permitted to call each of these curves a *spherical ellipse*.

2. Let the principal plane of the cone be drawn, and let us only consider the hemisphere and the sheet of the cone situated above this plane; this sheet will determine a spherical ellipse upon the surface of the sphere; the principal axis of the cone will penetrate the sphere in a point which will be the *centre* of this ellipse; for it will bisect every arc of a great circle passing through this point and included within the curve; since the arc of a great circle, included between

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two points on a sphere, measures the angle contained by two right lines drawn from the vertex of the cone to these two points.

Every arc of a great circle passing through the centre and terminated by the curve is a *diametral arc* of the curve.

The greatest and the least of the diametral arcs of the ellipse, which we shall call its *principal diametral arcs*, are contained in the planes of the greatest and least sections of the cone. These arcs terminate in four points of the ellipse, which may be called its *vertices*.

The two focal lines of the cone will penetrate the hemisphere, on which is traced the spherical ellipse which we are now considering, in two points which may be called the *foci* of the ellipse, on account of the similarity which may be shown to exist between their properties and those of the foci of plane ellipses.

Lastly, the two cyclic planes of the cone will intersect the same hemisphere in two great semi-circles, which will have the major axis of the cone for their common diameter. This major axis lies in the plane of the greatest diametral arc of the ellipse; and these two great semi-circles are perpendicular to the least diametral arc, and never meet the ellipse. In order to indicate their origin, let us call these two arcs the *cyclic arcs* of the spherical ellipse.

3. Now let us draw the plane of the least section of the cone. It is clear that this plane will divide the complete intersection of the cone and sphere into two equal parts, symmetrically placed with relation to this plane.

Let us consider the part of this intersection lying on one side of this plane: it will consist of two branches which will be halves of the two spherical ellipses; these two branches being symmetrical with respect to the diametral plane perpendicular to the principal axis of the cone, and receding more and more from this plane as the distance from their vertices increases, form, when considered together, a curve which may be called a *spherical hyperbola*. Its *centre* is the point where the major axis of the cone penetrates the hemisphere on which the curve is traced; its two *foci* are the points where the two focal lines of the cone penetrate this hemisphere. The curve has but two vertices, and its foci lie upon the arc of a great circle which joins them.

Lastly, the two cyclic planes of the cone intersect the hemisphere in two great semicircles, which pass through the centre of the hyperbola, and make equal angles with the arc which joins its two foci. These are the two *cyclic arcs* of the hyperbola.

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4. Again, let us consider the hemisphere lying on either side of the plane of the greatest section of the cone, that is, the plane containing the two focal lines.

On this hemisphere we shall have two halves of the two spherical ellipses. These two curves turn their concave parts towards each other, and approach towards the principal plane of the cone as the distance from their vertices increases.

These two branches taken together form a third species of spherical curve. This curve has a *centre*, which is the point of intersection of the sphere with the minor axis of the cone; it has four *foci* which are in the plane of the greatest section of the cone, and two *cyclic arcs* which lie between the two branches of the curve, and are perpendicular to the arc of the great circle which joins its two vertices.

The three curves which we have just been considering, are portions of the same curve which arises from the complete intersection of the sphere with a cone of the second degree, having its vertex at the centre of the sphere. We may, therefore, designate them all by the common name of *spherical conics*.

5. *The Spherical conics* possess a great number of properties, of which the most part are very remarkable.

It is to be observed, that *all these properties are double*, that is to say, to every proposition relative to the spherical conics, there always corresponds a second proposition relative to these same curves.

This results from the fact, that the properties of the cones of the second degree are double, as we have already proved. (See section 10 of the preceding Memoir.) But we may also prove this general principle by observing, that to any figure traced upon the surface of the sphere there always corresponds a second figure, which is the envelope of the arcs of great circles, whose planes are perpendicular to the radii of the sphere drawn to the different points of the first figure. Each property of the first figure has, therefore, always a corresponding property in the second figure; but if the first figure be a *conic*, the second will also be a *conic*, arising from the intersection of the sphere with the cone *supplementary* to that on which the first conic lies; which proves the principle that has been stated.

We may call the two conics *supplementary*, as well as the two cones; the cyclic arcs of either of them are in the diametral planes perpendicular to the diameters of the sphere passing through the foci of the other.

6. Among the numerous properties of the spherical conics

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there are two which have been already given by M. Magnus, of Berlin. (See Annales de Mathématiques, August, 1825.) They are the two following:

"The sum or the difference of the two radii vectores drawn from the two foci of a spherical conic to any point in it is constant.

"The two radii vectores drawn to any point of the conic make equal angles with the arc of a great circle touching the curve at that point."

7. It is evident that, by reference to the supplementary conic whose properties we have just stated, we might immediately deduce from these two propositions the following theorems which appear to us to possess an equal degree of interest:

"In every spherical conic there are two arcs of great circles such that the sum or the difference of the angles, which each arc of a great circle touching the conic makes with them, is constant.

"Every arc of a great circle, touching the conic and terminated by these two fixed arcs, is bisected at its point of contact with the conic."

The former of these two propositions shows that,

"The envelope of the bases of all the spherical triangles, which have a common vertical angle and the same area, is a spherical conic."

8. This latter, and the preceding theorem, are exactly analogous to the following well-known properties of the hyperbola:

"The envelope of the bases of all the plane triangles, which have a common vertical angle and the same area, is a hyperbola.

"The portion of any tangent to a hyperbola intercepted by the two asymptotes is bisected at the point of contact with the curve."

We shall find the same analogy between several other properties of the spherical conics and those of the hyperbola, so that the cyclic arcs of the spherical conics will be found to bear the same relation to those curves that its asymptotes do to the hyperbola.

As to the two theorems of M. Magnus, they bear a striking similarity to the known properties of the foci of the conic sections: the same may be observed of all the other properties of the foci of these curves.

In fact, we shall show in the last section of this Memoir, how the properties of the foci of the conic sections and those of the asymptotes of the hyperbola may be considered as consequences from those which we prove with relation to the foci and the cyclic arcs of the spherical conics.

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But as these properties of the foci of the spherical conics may be applied to all the plane conic sections, it might naturally be supposed that those of the cyclic arcs ought, in like manner, to have corresponding ones in every plane conic section; they therefore guide us to the discovery of a new class of general properties of the plane conic sections, of which those of the asymptotes of the hyperbola are only particular cases.

We shall make these new properties of the plane conic sections, the subject of a separate memoir, as we mean to couple with them the analogous properties of the surfaces of the second degree.

9. It is surprising that the elegance of M. Magnus's two propositions has not yet directed the attention of geometers to researches of this kind, and that the theory of the spherical conics has yet to be constructed, although, in consequence of the duality of all the propositions of spherical geometry, it admits perhaps of a wider extension than that of the plane conic sections.

We do not, indeed, pretend to lay before the reader a theory of these conics; but we propose merely to state a certain number of those among their properties which relate to the *foci* and to the *cyclic arcs*, and which result immediately from those which we have proved relative to the cones of the second degree. It is manifest that various other known properties of the cones of the second degree, would furnish, in like manner, and without difficulty, properties of the spherical conics, which ought to find a place in a treatise on these curves.

10. A spherical ellipse and a spherical hyperbola have each only two foci; but a complete spherical conic has four foci placed at the extremities of two diameters of the sphere. In the theorems relating to two foci, we must always be understood to refer to two foci assumed respectively on these two diameters, and not to two foci assumed upon the same diameter; for the two foci ought to belong to the two focal lines of the cone on which the conic is traced.

But it will be more simple to suppose, in all that follows, that the conic is a single spherical ellipse or hyperbola, and to consider upon the sphere only the hemisphere on which this ellipse or hyperbola is traced; by this means we shall avoid all ambiguity; an arc of a great circle touching the curve will have only one point of contact with it, whilst it would touch it in two points, were we to consider the complete conic; any two arcs of great circles will intersect in only one point, since we consider only one-half of the sphere; for the same reason three arcs of great circles will intersect two

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by two in only three points, and will form but a single spherical triangle.

As we shall only have to speak of arcs of great circles, we may be allowed, for the sake of brevity, to use merely the word *arc*; and it must be fully understood that we only mean arcs of great circles.

We shall call the angle formed by two arcs of great circles a *spherical angle*; their point of intersection will be the *vertex* of the angle. We shall call every arc of a great circle passing through the focus of a conic a *vector arc*.

The properties of the spherical conics which we are about to state being all immediate consequences from those of the cones of the second degree, which we have proved in the preceding Memoir, it will be sufficient to give the enunciations of them, indicating, in the case of each, the corresponding property of the cones of the second degree (*a*).

SECTION II.

PROPERTIES RELATING TO THE TWO CYCLIC ARCS OF A SPHERICAL CONIC; AND PROPERTIES RELATING TO ITS TWO FOCI.

11. The two theorems (20 *a*,) lead to the following:

Every arc of a great circle touching a spherical conic intersects the two cyclic arcs in two points, which are equally distant from the point of contact with this tangent arc.

The vector arcs, drawn from the two foci of a spherical conic to any point on the curve, make equal angles with the arc of a great circle touching the conic at that point.

12. Conversely,

If a curve traced upon a sphere be such that every arc of a great circle, touching the curve and terminated by two fixed arcs of great circles, is bisected at the point where it touches the curve, this curve is a spherical conic.

If a curve traced upon a sphere be such that the arcs of great circles, drawn from two fixed points to any point in the curve, make equal angles with the arc of a great circle touching the curve at that point, this curve is a spherical conic.

This follows from the two theorems (21 *a*.)

13. The two theorems (22 *a*,) lead to the following, of which the two in No. 11 are only particular cases:

Every arc of a great circle intersects a spherical conic in two points

The two vector arcs, drawn from the two foci of a spherical conic to

(*a*) We shall use numbers, followed by the letter *a*, in referring to the paragraphs of the preceding Memoir.

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which are equally distant from the points in which this arc respectively cuts the two cyclic arcs.

the point of intersection of two arcs touching the curve, respectively make equal angles with these tangent arcs.

14. Theorems (23 *a*,) lead to the following:

The planes of two arcs touching a spherical conic intersect the planes of the two cyclic arcs in four right lines, which are the generatrices of a right cone whose axis of revolution is perpendicular to the plane of the great circle passing through the two points of contact of the tangent arcs.

The planes of four vector arcs drawn from the two foci of a spherical conic to any two points of the curve will all touch the same right cone whose axis of revolution is the right line of intersection of the planes of the two arcs touching the conic at these two points.

15. Theorems (24 *a*,) lead to the following:

The sum or the difference of the angles which each arc touching a spherical conic makes with the two cyclic arcs is constant.

The sum or the difference of the vector arcs drawn from the two foci of a spherical conic to any point of the curve is constant.

16. Theorems (25 *a*,) lead to the following:

Every arc of a great circle touching a spherical conic intersects the two cyclic arcs in two points, such that the product of the trigonometric tangents of the semi-arcs lying between these points and the point of intersection of the two cyclic arcs is constant.

The vector arcs, drawn from the two foci of a spherical conic to any point of the curve, make, with the diametral arc which joins the two foci, two angles, such that the product of the trigonometric tangents of their halves is constant.

17. Theorems (26 *a*,) lead to the following:

In every spherical conic the product of the sines of the arcs of great circles drawn from any point of the curve at right angles with the two cyclic arcs is constant.

In every spherical conic the product of the sines of the arcs of great circles drawn from the two foci at right angles with any arc touching the curve is constant.

18. The two theorems (28 *a*, second and first columns,) respectively lead to the two following:

A spherical conic and its two cyclic arcs being given, if one extremity of an arc of 90° traverse either of the two cyclic arcs whilst the other extremity moves along the conic, this quadrantal arc will envelope a second spherical conic,

If from the foci of a spherical conic arcs be drawn perpendicular to the arcs touching the curve, their respective points of intersection with these tangent arcs will be upon a second spherical conic, which will have a double contact with the given

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which will have a double contact with the given one, and whose foci will be the extremities of the radii of the sphere perpendicular to the planes of the cyclic arcs of the given conic.

one, and whose cyclic arcs will be in the planes perpendicular to the radii of the sphere which pass through the two foci of the given conic.

19. The two theorems (31 *a*, second and first columns,) respectively lead to the following:

A spherical conic and one of its cyclic arcs being given, if between this arc and the curve, two arcs of 90° be inserted, and from their point of intersection a third quadrantal arc be drawn terminating in the arc which joins those two extremities of the two former ones which lie upon the conic, this extremity of the third arc will fall upon the second cyclic arc of the conic.

If from a focus of a spherical conic two arcs be drawn perpendicular to two arcs touching the curve, and if the feet of the two perpendicular arcs be joined by an arc of a great circle, the arc drawn at right angles with this latter from the point of concourse of the two tangent arcs will pass through the second focus of the curve.

20. Theorems (32 *a*,) lead to the following:

If two spherical conics have the same cyclic arcs, and a common tangent arc be drawn to them, the part of this arc intercepted between the two points of contact will be a quadrant.

If two spherical conics which have the same foci intersect, they will be at right angles to each other at each point of intersection.

SECTION III.

PROPERTIES OF THE SPHERICAL CONICS RELATING TO A SINGLE CYCLIC ARC; AND PROPERTIES RELATING TO A SINGLE FOCUS.

21. If through any point on the surface of a sphere two arcs of great circles be drawn touching a spherical conic, the arc of a great circle joining the two points of contact, may be called the *polar arc* of the point with relation to the conic; and conversely, this point will be called the *pole* of its polar arc.

From the properties of right lines and polar planes in cones of the second degree which have been stated (1 *a*,) it evidently follows, that

The polar arcs of all the points of any arc of a great circle, with relation to a spherical conic, pass all of them through the same point, which is the pole of that arc; and conversely,

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The poles of all the arcs of great circles passing through the same point, taken with relation to a spherical conic, are all of them upon the same arc of a great circle which is the polar arc of the fixed point.

To the polar arcs of its foci we give the name of *director arcs* of a spherical conic.

22. Hence, the two theorems (34, a,) lead to the following:

In every spherical conic, the sine of the angle which each tangent arc to the curve makes with a cyclic arc has a constant ratio to the sine of the distance of this tangent arc from the pole of the cyclic arc.

In every spherical conic, the ratio of the sines of the arcs which measure the distances of each point on the curve from a focus and from its corresponding director arc is constant.

23. Theorems (36, a,) lead to the following:

Every arc touching a spherical conic, and the arc drawn through its point of contact and through the pole of a cyclic arc of the conic, meet that cyclic arc in two points the distance between which is 90°.

The two vector arcs drawn from one focus of a spherical conic to any point of the curve and to the point where its tangent arc at the former point meets the director arc, are always at right angles.

24. Theorems (37, a,) lead to the following:

Two tangent arcs to a spherical conic, and the arc which joins their points of contact with the curve, intersect the cyclic arc in three points, the third of which bisects the distance between the first two.

The vector arcs drawn from a focus of a spherical conic to two points of the curve make equal angles with the vector arc drawn to the point of intersection of the two arcs touching the conic at those two points.

25. Theorems (38, a,) lead to the following:

Two tangent arcs to a spherical conic, and the arc passing through their point of intersection and through the pole of a cyclic arc, meet that arc in three points the third of which bisects the distance between the other two.

The vector arcs drawn from a focus of a spherical conic to two points on the curve make equal angles with the vector arc drawn to the point in which the arc joining the two points on the curve meets the director arc.

26. Theorems (39, a,) lead to the following:

The arc passing through the pole of a cyclic arc of a spherical conic, and through the point of intersection of two arcs touching the curve, and the arc passing through the two points of contact of these tangent arcs, meet the cyclic arc in two points, the distance between which is 90°.

The two vector arcs drawn from one focus of a spherical conic to the point of intersection of two arcs touching the curve, and to the point in which the arc passing through the two points of contact meets the director arc, are at right angles.

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The points in which the arc joining the two points of contact meets the cyclic arc, and the arc which joins its pole with the point of concourse of the two tangent arcs, are harmonic conjugates with relation to the two points of contact.

The two arcs passing through the point of concourse of the two tangent arcs, and passing, the one through the focus of the conic, and the other through the point in which the arc which joins the two points of contact meets the director arc, are harmonic conjugates with relation to the two tangent arcs.

27. The two theorems (40, *a*,) lead to the following:

If through a point assumed upon a cyclic arc of a spherical conic two arcs be drawn touching the conic, the arc joining the two points of contact will meet the cyclic arc in a point whose distance from the former is 90°.

If through a focus of a spherical conic a transversal arc be drawn, its pole with relation to the conic will be upon the director arc, and the arc drawn from the focus to this pole will be perpendicular to the transversal arc.

28. Theorems (41, *a*,) lead to the following:

When a spherical quadrilateral is inscribed in a spherical conic, the portion of a cyclic arc of the conic included between two adjacent sides of the quadrilateral is supplemental to the arc included between the two other sides.

When a spherical quadrilateral is circumscribed about a spherical conic, the angle between the two vector arcs drawn from one focus to two adjacent vertices of the quadrilateral is supplemental to the angle between the two vector arcs drawn to the two other vertices.

29. Theorems (42, *a*,) lead to the following:

If through two fixed points on a spherical conic two arcs be drawn which intersect in any third point of the curve, the segment which they will intercept upon a cyclic arc will be of invariable magnitude.

This segment will be a quadrant, if the arc which joins the two fixed points passes through the pole of the cyclic arc.

Two fixed arcs being drawn touching a spherical conic, and any third tangent arc intersecting the two former in two points, the vector arcs drawn from a focus of the conic to these two points will contain between them a constant angle.

This angle will be right, if the point of concourse of the two fixed tangent arcs be upon the director arc corresponding to the focus.

30. The two theorems (43, *a*,) lead to the following:

If through the two vertices which are at the extremities of the least diametral arc of a spherical ellipse two arcs be drawn intersecting in any third point of the curve, the segment intercepted between these two arcs upon a cyclic arc will be a quadrant.

Every arc touching a spherical conic cuts the arcs touching the curve at the two vertices which are at the extremities of its greatest diametral arc in two points such that the two vector arcs drawn from a focus to these two points are at right angles.

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31. Theorems (44, *a*,) lead to the following:

If upon a cyclic arc of a spherical conic a segment of given magnitude be arbitrarily assumed, and through one of its extremities and a fixed point of the curve an arc be drawn meeting the curve in a second point, the arc drawn through this second point and through the other extremity of the segment will pass through a fixed point on the conic.

If round a focus of a spherical conic, as vertex, a spherical angle of invariable magnitude be made to turn, and through the point where one of its sides meets a fixed arc touching the curve, a second tangent arc be drawn, this arc will meet the second side of the angle in a point the geometric locus of which will be an arc touching the conic.

32. Theorems (45, *a*,) lead to the following:

If through a point assumed arbitrarily on a cyclic arc of a spherical conic two arcs be drawn touching the curve, the sum of the trigonometric co-tangents of the angles which they make with the cyclic arc will be constant.

If through a focus of a spherical conic an arc be drawn arbitrarily meeting it in two points, the sum of the trigonometric co-tangents of the arcs lying between the foci and these two points will be constant.

SECTION IV.

GEOMETRIC LOCI RELATING TO THE CYCLIC ARCS AND TO THE FOCI OF THE SPHERICAL CONICS.

33. Theorems (46 *a*,) lead to the following:

If the sides of a spherical angle of variable magnitude pass always through two fixed points on the surface of a sphere, whilst the segment intercepted between its sides upon an arc of a given great circle is of a constant length, the vertex of this angle will generate a spherical conic which will have the fixed arc for a cyclic arc, and which will pass through the two fixed points.

Two fixed arcs and a point being given on a sphere, if round this point as vertex a spherical angle of invariable magnitude be made to turn, and if the two points in which its sides respectively meet the two fixed arcs be joined by an arc of a great circle, this arc will envelope a spherical conic which will have the fixed point for a focus, and which will touch the two fixed arcs.

34. Theorems (47 *a*,) lead to the following:

If two tangent arcs be drawn to a spherical conic so that the segment intercepted between them upon a cyclic arc of the conic may be of a

If an angle of invariable magnitude be made to turn round a focus of a spherical conic as vertex, the arc joining the two points in which its

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constant length, the geometric locus of the point of concourse of these two arcs will be a second spherical conic.

The arc joining the two points of contact of these two arcs with the given conic will envelope a third conic.

The cyclic arc in question will be a cyclic arc of the two new conics, and this arc will have the same pole with relation to the three conics.

sides meet the curve will envelope a second conic.

The tangent arcs to the given conic at these two points will intersect upon a third conic.

The focus in question will also be a focus of the two new conics, and the corresponding director arc will be the same in the three conics.

35. Theorems (48, a,) lead to the following:

If round a fixed point assumed on a spherical conic a spherical angle of variable magnitude be made to turn, whose sides intercept upon a cyclic arc of the curve a segment of a constant length, the arc joining the two points in which the sides of this angle meet the conic will envelope a second conic; the cyclic arc upon which the segments are measured will be a cyclic arc of the new conic, and it will have the same pole in the two curves.

If round a focus of a spherical conic as vertex a spherical angle of invariable magnitude be to turn, and through the two points in which its sides meet a fixed arc touching the conic two arcs be drawn touching the curve, the point of concourse of these two arcs will generate a second conic; the focus of the given conic will also be a focus of the new conic, and the corresponding director arc will be the same in the two curves.

36. Theorems (49, a,) lead to the following:

A spherical conic and a fixed arc arbitrarily drawn being given, if two tangent arcs to the conic be drawn so that the segment intercepted between them on the given arc may be a quadrant, the point of concourse of these two arcs will generate a second conic which will have the given arc for a cyclic arc.

This arc will have the same pole in the two conics.

37. *If the fixed arc be one of the principal diametral arcs of the conic, the theorem may be thus stated:*

If a variable spherical angle, circumscribed about a spherical conic, move so that the segment intercepted between its sides upon a principal diametral arc of the conic may always be a quadrant, the vertex of this angle will generate a small circle of the sphere.

A spherical conic and a fixed point arbitrarily assumed on the sphere being given, if round this point as vertex a right spherical angle be made to turn, and if the points in which its sides meet the conic, taken two by two, be joined by four arcs, these four arcs will envelope a second conic of which the fixed point will be a focus.

This point will have the same polar arc in the two curves.

If the fixed point be the centre of the conic, the theorem may be thus stated:

If a right spherical angle turn round the centre of a spherical conic as vertex, the arc joining the points in which its two sides meet the curve will envelope a small circle of the sphere.

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This is also a consequence from the two theorems (50, *a*.)

38. The two theorems (51, *a*,) lead to the following:

If round two fixed points on a spherical conic two arcs be made to turn intersecting in any third point of the curve, these arcs will respectively meet the two cyclic arcs of the conic in two points, and the arc joining these two points will envelope a spherical conic touching these two cyclic arcs.

Two fixed tangent arcs being drawn to a spherical conic, any third tangent arc will intersect them in two points, and the arcs respectively drawn through these points and through the two foci of the conic will intersect in a point the geometric locus of which will be a spherical conic passing through the two foci of the given one.

39. Theorems (52, *a*,) lead to the following:

If round two fixed points two arcs be made to turn, containing between them a right angle, their point of intersection will generate a spherical conic passing through the two fixed points, and whose cyclic arcs will be in the two planes perpendicular to the radii of the sphere drawn to the two fixed points.

If the extremities of an arc of 90° move along the sides of any given spherical angle, this moveable arc will envelope a spherical conic which will touch the two sides of the angle, and whose foci will be the extremities of the radii of the sphere perpendicular to the planes of these sides.

40. Theorems (53, *a*,) lead to the following:

Two fixed arcs being given upon a sphere, if a point be sought such that the product of the sines of its distances from the two fixed arcs may be constant, the geometric locus of this point will be a spherical conic whose cyclic arcs will be the two given arcs.

Two fixed points being given upon a sphere, if an arc be drawn such that the product of the sines of its distances from these two points may be constant, this arc will envelope a spherical conic whose foci will be the two given points.

41. Theorems (54, *a*,) lead to the following:

An arc and a point being given upon a sphere, if an arc be sought such that the sine of the angle which it makes with the given arc, and the sine of its distance from the given point, may have a constant ratio, this arc will envelope a conic which will have the given arc for a cyclic arc; and the given point will be, with relation to this conic, the pole of that cyclic arc.

A point and an arc being given upon a sphere, if a point be sought such that the sines of its distances from the given point and arc may have a constant ratio, the geometric locus of this point will be a conic of which the given point will be a focus; and the given arc will be the director arc corresponding to that focus.

42. If in the second theorem the ratio of the sines be one of equality, we infer from it, that

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The spherical curve every point of which is equi-distant from a given point and from a given arc of a great circle is a conic having this point and arc for a focus and its corresponding director arc.

SECTION V.

PROBLEMS RELATING TO THE CYCLIC ARCS AND TO THE FOCI OF THE SPHERICAL CONICS; AND GENERAL PROPERTIES OF SPHERICAL TRIANGLES AND QUADRILATERALS.

43. When a cyclic arc of a spherical conic is given, only three other conditions are required in order to determine the curve.

When a focus of a spherical conic is given, only three other conditions are required in order to determine the curve.

This is a consequence of what we have said with reference to cones of the second degree (56, *a*.)

44. Theorems (57, *a*,) lead to the following:

A cyclic arc and two points of a spherical conic being given, the pole of this cyclic arc lies upon the arc which passes through the two following points:

1. The point which is, with relation to the two given points, the harmonic conjugate of that in which the arc joining these two points meets the given cyclic arc.

2. The point on the cyclic arc which is 90° distant from that in which this arc meets the arc joining the two given points.

A focus and two tangent arcs to a spherical conic being given, the director arc corresponding to that focus passes through the point of intersection of the two following arcs:

1. The arc which passes through the point of concourse of the two given tangents, and which is the harmonic conjugate, with relation to these two arcs, of the arc drawn through this point of concourse and through the given focus.

2. The arc drawn through this focus perpendicular to this latter arc joining the focus with the point of concourse of the two tangent arcs.

45. Theorems (58, *a*,) lead to the following:

A cyclic arc and two tangent arcs to a spherical conic being given,

The pole of that cyclic arc lies upon the arc drawn through the point of concourse of the two tangent arcs and through the middle of the arc (or the supplement of the arc) intercepted upon this cyclic

A focus and two points on a spherical conic being given,

The director arc corresponding to that focus passes through the point in which the arc joining the two given points meets the vector arc which bisects the angle (or the supplement of the angle) contained

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arc between the two given tangent arcs.

between the two vector arcs drawn from the given focus to the two points on the conic.

46. The two theorems (44,) respectively contain the solutions of the two following problems:

Problem.—Given three points and a cyclic arc of a spherical conic, to determine the pole of this cyclic arc.

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the director arc corresponding to this focus.

47. The two theorems (45,) in like manner enable us to resolve the two following problems, each of which admits of four solutions:

Problem.—Given three tangent arcs and a cyclic arc of a spherical conic, to determine the pole of this cyclic arc.

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the director arc corresponding to that focus.

48. We have just seen (46,) that

A spherical triangle being given, and also any arc of a great circle, this arc may be considered as a cyclic arc of a spherical conic passing through the three vertices of the triangle.

A spherical triangle being given, and also a fixed point upon the sphere, this point may be considered as the focus of a spherical conic touching the three sides of the triangle.

This remark will aid us in the proof of some general properties of spherical triangles and quadrilaterals.

49. Theorems (13,) lead, as appears from what we have just said, to the two following properties of spherical triangles:

If a spherical triangle and a transversal arc be given, and upon each side of the triangle a point be assumed whose distance from one extremity of that side is equal to the distance of the other extremity from the point in which the transversal arc meets the side, the three points thus determined upon the three sides of the triangle will lie upon the same arc of a great circle.

A spherical triangle being given, if through a fixed point an arc be drawn to each vertex of the triangle, and through that vertex a second arc be drawn which makes with one of the adjacent sides of the triangle an angle equal to that which the first arc makes with the other side, the three arcs thus drawn will pass through the same point.

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This arc and the given transversal arc will be the cyclic arcs of a conic circumscribed about the spherical triangle.

This point and the given point will be the foci of a spherical conic inscribed in the given triangle.

50. *This theorem furnishes the solution of the following*

Problem.—Given three points and a cyclic arc of a spherical conic, to determine the second cyclic arc of this curve.

This theorem furnishes the solution of the following

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the second focus of this curve.

51. Theorems (18) lead to the two following properties of spherical triangles:

A spherical triangle and a transversal arc being given, if upon this arc three points be assumed which are respectively distant by 90° from the three vertices of the triangle, and these points be connected with the three vertices by three arcs, if we now describe a conic touching these last three arcs and having for its focus the extremity of the radius of the sphere perpendicular to the plane of the given transversal arc, and through the vertices of the given triangle draw three new arcs touching this conic, and if we assume upon these arcs three points respectively distant by 90° from the vertices, these three points will be upon the same arc of a great circle.

If from a point assumed arbitrarily upon a sphere three arcs be drawn perpendicular to the three sides of a spherical triangle, and through the feet of these perpendiculars a spherical conic be made to pass, which has for a cyclic arc the great circle contained in the plane perpendicular to the radius of the sphere drawn to the given point, this conic will meet the three sides of the triangle in three new points such that the arcs drawn through these points and respectively perpendicular to the three sides will pass through the same point.

This arc will be in the plane perpendicular to the radius of the sphere drawn to the second focus of the conic.

This point will be the extremity of the radius perpendicular to the plane of the second cyclic arc of the conic.

52. Theorems (19) lead to the two following properties of spherical triangles:

A triangle and an arc of a great circle being traced upon a sphere, if through each vertex of the triangle an arc of 90° be drawn terminated by the given arc, the three arcs thus drawn will form a second spherical triangle; if through the vertices of this new triangle three arcs of 90° be drawn, respectively terminated by the three opposite sides of the first triangle, the extremities of these three arcs will lie upon the same arc of a great circle.

If from a point assumed upon a sphere arcs be drawn perpendicular to the three sides of a spherical triangle, their feet will be the three vertices of a new triangle inscribed in the first; and if from the vertices of the first triangle arcs be drawn respectively perpendicular to the opposite sides of the second triangle, these three arcs will pass through the same point.

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53. The two theorems (67 *a*,) lead to the following:

If about a spherical conic any number of spherical triangles be circumscribed, and as many spherical conics be circumscribed about these triangles, having all of them for a common cyclic arc a fixed tangent arc to the given conic, all these curves will pass through the same point.

If in a spherical conic any number of spherical triangles be inscribed, and as many spherical conics be inscribed in these triangles, having all of them for a common focus, a fixed point on the given conic, all these curves will touch the same arc of a great circle.

54. From the two theorems (68 *a*,) we deduce the following:

A spherical conic being given, if any spherical triangle be circumscribed about it, and if upon a fixed arc touching the conic three points be assumed such that the arcs respectively drawn from these points to the three vertices of the triangle may be quadrants; and if we further suppose a conic to be described touching these three arcs, and having for its focus the extremity of the radius of the sphere perpendicular to the plane of the fixed arc, this new conic will always touch the same arc of a great circle, whatever be the triangle circumscribed about the given conic.

If from a fixed point assumed upon a spherical conic, three arcs be drawn perpendicular to the sides of a spherical triangle inscribed in the conic, and if through the feet of these perpendiculars a spherical conic be made to pass, having for a cyclic arc the arc of a great circle lying in the plane perpendicular to the radius of the sphere drawn to the fixed point on the given conic, this new conic will always pass through a fixed point, whatever be the triangle inscribed in the given conic.

55. The two theorems (69 *a*,) lead to the two following general properties of spherical quadrilaterals:

A spherical quadrilateral being given, its sides, taken three by three, will form four triangles; if through the vertices of each of these triangles three quadrantal arcs be drawn, terminated by the same given arc of a great circle, and if a conic be described, touching these three arcs, and having for a focus the extremity of the radius of the sphere perpendicular to the plane of the given great circle, the four conics thus determined will all touch the same arc of a great circle.

A spherical quadrilateral being given, its four vertices, taken three by three, will determine four triangles; if from a fixed point arcs be drawn perpendicular to the three sides of each of these triangles, and if through the three points in which these arcs respectively meet the sides, a spherical conic be made to pass, having one of its cyclic arcs in the plane perpendicular to the radius of the sphere which is drawn to the given fixed point, the four conics thus determined will all pass through the same point.

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SECTION VI.

ORGANIC DESCRIPTION OF THE SPHERICAL CONICS.

56. The two theorems (71 *a*,) relating to the description of the cones of the second degree, respectively lead to the two following:

If any two spherical angles, each of invariable magnitude, turn round two fixed points as vertices, so that two of their sides intersect on a given fixed arc, the point of intersection of their two other sides will generate a spherical conic which will pass through the two fixed vertices of the moveable angles.

If along two given fixed arcs any two segments, each of invariable magnitude, be made to move, so that the arc joining two of their extremities may turn round a fixed point, the arc joining their two other extremities will envelope a conic which will touch the two fixed arcs along which the two segments move.

The first of these two theorems is exactly analogous to that of Newton, relative to the organic description of the plane conic sections.

57. *Problem.*—Given five points of a spherical conic, to determine all the other points of the curve by the movement of two spherical angles round their vertices.

Let A, B, C, D, E, *be the five given points; let us take the first two* A, B, *for the vertices of the two moveable angles, and these angles will be* (CAB), (CBA).

Now let these angles be turned round their vertices A, B, *so that their sides* CA, CB, *may pass at the same time through the point* D, *and again through the point* E; *their two other sides, which originally coincided with the arc* AB, *will successively intersect in two points* D′, E′.

Problem.—Given five tangent arcs to a spherical conic, to determine all the other tangent arcs of the curve by the movement of two arcs along the circumferences of two great circles.

Let A, B, C, D, E, *be the five given arcs; let us take the circumferences of the great circles to which the first two,* A, B, *belong, for those along which the two moveable arcs are to be measured; the third arc* C, *produced if necessary, will meet the first two* A, B, *in two points, and the arcs included between these two points and the point of intersection of* A *and* B *will be the two moveable arcs.*

Now let these two arcs be moved along their circumferences, so that their extremities, which before were placed upon the arc C, *may be upon the arc* D, *and again upon the arc* E; *their two other extremities, which originally coincided with the point of intersection of the two arcs* A, B, *will successively determine two arcs* D′, E′.

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Let the point of intersection of the same two sides traverse the arc of a great circle determined by the two points D', E'; *the point of intersection of the two other sides will generate the required conic, as appears from the preceding theorem.*

Let the arc joining the same two extremities turn round the point of intersection of these two arcs D', E'; *the arc joining the two other extremities of the two moveable arcs will envelope the required conic, as appears from the preceding theorem.*

SECTION VII.

PROPERTIES OF THE PLANE CONIC SECTIONS DEDUCED AS CONSEQUENCES FROM THOSE OF THE SPHERICAL CONICS.

58. We have stated (8), that from the propositions relative to the spherical conics, contained in this memoir, we might deduce a very great number of the properties of the foci of the plane conic sections, and some properties of the asymptotes of the hyperbola.

For this purpose, it is sufficient to suppose that the centre of the sphere recedes to an infinite distance upon the radius which passes through the centre of the spherical conic. This curve will degenerate into a plane conic which will be an ellipse or a hyperbola; and the properties of the foci of the spherical conics will become those of the plane conics.

In the case where the conic becomes a hyperbola, the cyclic arcs will become two fixed right lines drawn through the centre of the curve; and the properties of the cyclic arcs will apply to these two right lines, which each of these properties leads us to recognize as the asymptotes of the hyperbola.

59. We shall state the various theorems which may be deduced in this way from the properties of the spherical conics; at the end of each we shall refer to the number of the theorem from which it is a consequence. First, we shall give those which relate to the foci, and afterwards those which relate to the asymptotes.

These two classes of theorems which are thus found to have a common origin, present a remarkable connexion between the properties of the foci and those of the asymptotes; properties so different both in their statement and in the proofs usually given of them. When discussing the new properties of the conic sections of which we have spoken (8), we mean to shew how close a relation subsists in other respects between the foci of the conic sections and the asymptotes of the hyperbola.

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I.

GENERAL PROPERTIES OF THE TWO FOCI OF THE PLANE CONIC SECTIONS CONSIDERED SIMULTANEOUSLY.

1. The radii vectores drawn from the two foci to any point of the conic section make equal angles with the tangent at that point (11).

2. Conversely: If a curve be such that the radii vectores drawn from two fixed points to each point on it make equal angles with the tangent at that point, the curve is a conic section (12).

3. The two radii vectores drawn from the two foci of a conic section to the point of concourse of two tangents respectively make equal angles with those tangents (13).

4. The four radii vectores drawn from the two foci of a conic section to any two points on it are tangents to the same circle, whose centre is the point of concourse of the tangents to the conic section at those two points (14).

5. The sum or the difference of the two radii vectores drawn from the two foci of a conic section to any point on it is constant (15).

6. The product of the trigonometric tangents of the semi-angles, which the two radii vectores, drawn from the two foci of the conic section to any point on it, make with its major axis, is constant (16).

7. The rectangle under the perpendiculars let fall from the two foci of a conic section on each tangent to the curve is constant (17).

8. The locus of the feet of the perpendiculars, let fall from the two foci of a conic section upon its tangents, is a circle (18).

9. If from a focus of a conic section perpendiculars be let fall upon two tangents to the curve, and if a right line be drawn joining their feet, the perpendicular drawn to this right line from the point of concourse of the two tangents will pass through the second focus of the curve (19).

10. If two conic sections which have the same foci cut one another, they are at right angles to each other at each point of intersection (20).

II.

PROPERTIES OF THE PLANE CONIC SECTIONS RELATING TO A SINGLE FOCUS.

1. The ratio of the distances of each point on a conic section from a focus and from the corresponding directrix is constant (22).

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2. The radii vectores drawn from a focus of a conic section to a point on the curve, and to the point where the tangent at that former point meets the directrix, are at right angles (23).

3. The radii vectores drawn from a focus of a conic section to two points on the curve make equal angles with the radius vector drawn to the point of concourse of the two tangents at those points (24).

4. The two radii vectores, drawn from a focus of a conic section to two points on the curve, make equal angles with the radius vector drawn to the point in which the chord joining the two points on the curve meets the directrix (25).

5. The radius vector drawn from a focus of a conic section to the point of concourse of two tangents to the curve, and the radius vector drawn to the point in which the chord joining the two points of contact meets the directrix, are at right angles.

The two right lines drawn through the point of concourse of the two tangents, and passing, the one through the focus, and the other through the point in which the chord joining the two points of contact meets the directrix, are harmonic conjugates with relation to the two tangents (26).

6. If a transversal be drawn through the focus of a conic section, its pole, with relation to the curve, will be upon the directrix, and the radius vector drawn from the focus to this pole will be perpendicular to the transversal (27).

7. When a quadrilateral is circumscribed about a conic section, two opposite sides of the quadrilateral subtend at either focus two angles supplemental one to the other (28).

8. Two fixed tangents being drawn to a conic section, the part of another moveable tangent intercepted between the two former ones will subtend at either focus an angle of invariable magnitude.

This angle will be right, if the point of concourse of the two fixed tangents be upon the directrix (29).

9. The portion of any tangent of a conic section intercepted between the two tangents drawn at the extremities of the major axis subtends a right angle at either focus (30).

10. A conic section and a fixed tangent to it being given, if round either focus, as vertex, an angle of invariable magnitude be made to turn, and through the point in which one of its sides meets the fixed tangent we draw a second tangent to the curve, this second tangent will meet the second side of the angle in a point, the geometric locus of which will be a tangent to the conic section (31).

11. Any chord of a conic section passing through a focus

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is divided at that point into two parts the sum of the reciprocals of which is constant (32).

III.

GEOMETRIC LOCI RELATING TO THE FOCI OF THE PLANE CONIC SECTIONS.

1. If an angle of invariable magnitude be made to turn round a fixed point as vertex, and if a right line be drawn joining the two points in which its two sides respectively meet two given right lines, this right line will envelope a conic section whose focus will be the vertex of the moveable angle, and which will touch the two given right lines (33).

2. If an angle of invariable magnitude be made to turn round the focus of a conic section as vertex, the chord which it will subtend will envelope a second conic section.

The tangents to the given conic section at the extremities of this chord will intersect in a point the geometric locus of which will be a third conic section.

These two new conic sections will have the same focus as the given conic section, and the same corresponding directrix (34).

3. If an angle of given magnitude be made to turn round the focus of a conic section as vertex, and through the points in which its sides meet a tangent to the curve, two new tangents be drawn, their point of concourse will generate a second conic section, of which the fixed vertex of the moveable angle will be a focus, and the corresponding directrix will be that of the given conic section (35).

4. If in the plane of a conic section a right angle be made to turn round a fixed point as vertex, the chord subtended in the conic section by the sides of this angle will envelope a second conic section, one of whose foci will be at the fixed point, and the corresponding directrix will be the polar of this point with relation to the given conic section (36).

If the vertex of the moveable angle be the centre of the given conic section, the second conic section will be a circle(37).

5. A conic section and two fixed tangents being given, if any third tangent be drawn, and through the points in which it meets the two fixed tangents, two right lines be drawn respectively passing through the two foci of the curve, these two right lines will intersect in a point, the geometric locus of which will be a conic section passing through the two foci of the given one (38).

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6. Two fixed points being given, if a right line be drawn, such that the rectangle under its distances from the two given fixed points shall be constant, this right line, and all the others determined in like manner, will envelope a conic section whose foci will be the two fixed points (40).

7. A point and a right line being given, the geometric locus of a point whose distances from the given point and right line are to each other in a constant ratio, is a conic section in which the given point is a focus, and the given right line is the corresponding directrix (41).

IV.

PROBLEMS RELATING TO THE FOCI OF THE PLANE CONIC SECTIONS, AND GENERAL PROPERTIES OF PLANE TRIANGLES AND QUDRILATERALS.

1. When one of the foci of a conic section is given, only three other conditions are required to determine this curve. (43).

2. A focus and two tangents of a conic section being given, the directrix corresponding to that focus passes through the point of intersection of the two following right lines:

a. The right line which passes through the point of concourse of the two given tangents, and is the harmonic conjugate, with relation to these two tangents, of the right line drawn from this point of concourse to the focus of the curve:

b. The right line drawn through the focus perpendicular to the right line joining this focus with the point of concourse of the two given tangents (44).

3. A focus and two points on a conic section being given, the directrix corresponding to that focus passes through the point in which the right line joining the two given points meets the right line which bisects the angle, or the supplement of the angle, contained between the two radii vectores drawn from the focus to the two given points (45).

4. The last theorem but one furnishes the solution of the following problem:

Given three tangents and one of the foci of a conic section, to determine the directrix corresponding to that focus.

5. The last theorem enables us to resolve the following problem, which admits of four solutions:

Given three points and one of the foci of a conic section, to determine the directrix corresponding to that focus.

6. Three right lines being drawn from a fixed point to the three vertices of a triangle, if through each vertex a new right

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line be drawn, making with one of the two sides adjacent to that vertex an angle equal to that which the former right line makes with the other side, the three right lines thus drawn will pass through the same point.

This point and that from which the three former right lines were drawn, will be the foci of a conic section inscribed in the given triangle (49).

7. This theorem furnishes a solution of the following problem:

Given three tangents and one of the foci of a conic section, to determine the other focus.

8. If from a point assumed arbitrarily in the plane of a triangle three perpendiculars be let fall upon its sides, and a circle be described passing through their feet, the perpendiculars drawn to the sides of the triangle at the three new points in which this circle meets them, will all pass through the same point (51).

9. If from a point assumed in the plane of a triangle three perpendiculars be let fall upon its sides, their feet will be the vertices of a second triangle inscribed in the former; and if through the vertices of the first triangle right lines be drawn respectively perpendicular to the opposite sides of the second, these three right lines will pass through the same point (52).

The feet of the perpendiculars let fall from this new point upon the sides of the given triangle, and those of the three former perpendiculars, will be six points lying on the circumference of the same circle.

10. If any number of triangles be inscribed in a conic section, and as many conic sections be again inscribed in these triangles, having all of them for their common focus a fixed point on the given conic section, all these curves will touch the same right line (53).

11. If from a point assumed upon a conic section perpendiculars be let fall upon the sides of a triangle inscribed in the conic section, the circle described through the feet of these perpendiculars will pass through a fixed point, whatever be the triangle inscribed in the conic section (54).

12. The four vertices of a quadrilateral, taken three by three, determine four triangles; if from any point perpendiculars be let fall upon the sides of each triangle, and a circle be described passing through their feet, the four circles thus determined will pass through the same point (55).

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V.

PROPERTIES OF THE TWO ASYMPTOTES OF THE HYPERBOLA CONSIDERED SIMULTANEOUSLY.

1. Every tangent to a hyperbola meets the asymptotes in two points, which are equally distant from the point of contact (11).

2. Conversely: If a curve be such that the portion of each tangent intercepted between two given right lines is bisected at the point of contact, this curve is a hyperbola whose asymptotes are the two given right lines (12).

3. The portions of any secant intercepted between the hyperbola and its asymptotes are equal (13).

4. Every tangent to a hyperbola meets the asymptotes in two points, the rectangle under the distances of which from the centre of the curve is constant (16).

5. In every hyperbola the rectangle under the distances of any point on the curve from the two asymptotes is constant (17).

VI.

PROPERTIES OF THE HYPERBOLA RELATING TO A SINGLE ASYMPTOTE.

1. Two tangents to a hyperbola, and the right line joining the points of contact, meet an asymptote in three points, the third of which bisects the distance between the first two (24).

2. If the two sides of a variable angle, whose vertex traverses a hyperbola, pass through two fixed points on the curve, the segment intercepted between the sides of this angle upon an asymptote will be of a constant length (29).

3. If on either asymptote of a hyperbola a portion of given length be arbitrarily assumed, and through one of its extremities and a fixed point on the curve a right line be drawn meeting the curve in a second point, the right line joining this second point with the second extremity of the portion assumed on the asymptote will turn round a fixed point on the hyperbola (31).

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VII.

GEOMETRIC LOCI RELATING TO THE ASYMPTOTES OF THE HYPERBOLA.

1. If the two sides of an angle of variable magnitude pass always through two fixed points, and intercept upon a given right line a segment of a constant length, the vertex of this angle will generate a hyperbola, which will pass through the two fixed points, and which will have the given right line for an asymptote (33).

2. If two tangents to a hyperbola intercept between them on one of the asymptotes a segment of a constant length, the geometric locus of their point of concourse will be a second hyperbola.

The chord joining the two points of contact will envelope a third hyperbola.

The asymptote on which are measured the segments intercepted between the tangents will be an asymptote of the two new hyperbolas (34).

3. If an angle of variable magnitude be made to turn round a fixed point on a hyperbola as vertex, intercepting on an asymptote a segment of a constant length, the chord subtended by this angle will envelope a second hyperbola, to which the right line on which the intercepted segments are measured will also be an asymptote (35).

4. If the sides of a variable angle pass always through two fixed points on a hyperbola, whilst its vertex traverses the curve, the sides of this angle will respectively meet the two asymptotes in two points, and the right line joining these two points will envelope a conic section touching the two asymptotes of the hyperbola (38).

5. Two right lines being given, the geometric locus of the point, the rectangle under whose distances from the two right lines is constant, will be a hyperbola of which the two right lines are the asymptotes (40).

VIII.

GEOMETRIC LOCI RELATING TO ANY CONIC SECTIONS.

The first three of the five preceding theorems give rise to new theorems, by means of the method of transforming geometrical relations which we have explained in a preceding Memoir; these theorems, which relate to any conic section, are the following:

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"1. If an angle of variable magnitude be made to turn round a fixed point as vertex, so that the segment which it intercepts upon a fixed axis may be of a constant length, the right line joining the two points in which the sides of this angle respectively meet two given right lines will envelope a conic section, which will touch these two right lines, and which will pass through the vertex of the moveable angle.

"The tangent to the curve at this point will be parallel to the axis on which the segments are measured."

This proposition furnishes the solution of the following problem:

"Given four tangents to a conic section, and the point of contact of one of these tangents, to determine all the other tangents of the curve, by the continued movement of a segment of a constant length along a fixed right line."

"2. If an angle of variable magnitude be made to turn round a point on a conic section as vertex, so that the segment which it intercepts upon a fixed axis parallel to the tangent at that point shall be of a constant length, the chord subtended by this angle in the conic section will envelope a second conic section.

"The point of concourse of the two tangents to the first conic section at the extremities of this chord will generate a third conic section.

"These two new curves will touch the given one at the vertex of the moveable angle."

"3. If an angle of variable magnitude be made to turn round a fixed point on a conic section as vertex, whilst its sides intercept a segment of a constant length upon a parallel to the right line touching the curve at that point, and if through the points in which the sides of this angle meet a fixed tangent to the curve two new tangents be drawn, their point of concourse will generate a conic section, which will touch the given one at the vertex of the moveable angle."

IX.

PROBLEM RELATING TO THE ASYMPTOTES OF THE HYPERBOLA, AND GENERAL PROPERTIES OF TRIANGLES.

1. One of the asymptotes of a hyperbola being given, only three other conditions are required to determine the curve (43).

2. If any transversal be drawn in the plane of a triangle, and on each side a point be assumed, whose distance from one of the two extremities of that side is equal to the distance of the other extremity from the point in which the transversal

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meets that side, the three points thus assumed will be in the same right line (49).

3. This theorem furnishes the solution of the following problem:

Given three points and one of the asymptotes of a hyperbola, to find the second asymptote.

4. If any number of triangles be circumscribed about a hyperbola, and if we conceive as many hyperbolas to be circumscribed about these triangles, having all of them a tangent of the given hyperbola for their common asymptote, all these curves will pass through the same point (53).

5. By means of the method of transforming geometrical relations which we have already employed (No. VIII. in this section) the preceding theorem (2) gives rise to the following general property of triangles:

"If through each vertex of a triangle two right lines be drawn, of which the first passes through a given fixed point, and the second is such that the angles, which these two right lines respectively make with the two sides of the triangle adjacent to the vertex, intercept equal segments upon a fixed transversal, the three right lines thus determined will pass through the same point."

X.

ORGANIC DESCRIPTION OF THE PLANE CONIC SECTIONS.

The two theorems (56) relative to the description of the spherical conics lead to the two following:

1. If two angles of given magnitudes turn round two fixed points as vertices, so that the point of concourse of two of their sides traverses a right line, the point of concourse of their two other sides will generate a conic section, which will pass through the two fixed points.

2. If along two fixed right lines two segments of given lengths be made to move, so that two of their extremities are always in the same right line with a fixed point, the right line joining their two other extremities will envelope a conic section, which will touch the two fixed right lines.

The first of these two theorems is that of Newton, and enables us to describe the conic section by points; the second, which is new, furnishes a very simple construction of the tangents of the conic sections, as we shall presently see in the solution of the following problem:

3. *Problem.* Given five tangents of a conic section, to determine all its other tangents by the movement of two rectilineal segments along two fixed right lines.

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Let A, B, C, D, E, be the five given right lines; let us take the first two A, B, as the two fixed right lines along which the two segments are to move, and for the segments themselves let us take the distances of the point of concourse of these two right lines from the points in which they are intersected by the third right line C. These two segments must be made to move respectively along the two right lines A, B, so that their extremities which before lay upon the right line C, may fall upon the right line D, and again upon the right line E; their two other extremities, which at first coincided with the point of intersection of the two right lines A, B, will successively determine two right lines D′, E′.

Let the two segments move so that these same two extremities may always be in the same right line with the point of intersection of the two right lines D′, E′; the right line joining the two other extremities of the two segments will assume all the positions of the tangents to the required conic section.

4. It is unnecessary to explain the construction by which, as a consequence from Newton's theorem, we determine all the points of a conic section subject to the condition of passing through five given points; it would be a mere repetition of what we have already said in the solution of the same question with respect to the spherical conics (57).

5. Before closing this Memoir, we may observe that the theorem (2), which enables us to construct the tangents of the conic sections, gives rise, by means of our method of transforming geometrical relations, to another theorem available in the construction of conic sections by points, and which, for this purpose, might take the place of Newton's theorem.

This new theorem may be thus stated:

"If two angles intercepting segments of constant lengths upon a fixed axis turn round two fixed points as vertices, so that two of their sides always intersect upon a given right line, the point of concourse of their two other sides will generate a conic section passing through the two fixed points."

This theorem may be employed in the same way that Maclaurin, in his Organic Geometry, made use of Newton's theorem.

6. In order to complete this chapter on the organic construction of conic sections we ought to add, that Newton's theorem also gives rise to a theorem relating to the construction of the tangents of the conic section by the movement of two angles of constant magnitudes. This theorem may be obtained by a polar transformation, a circle being used as the auxiliary conic, as M. Poncelet has pointed out in his Memoire Sur la Théorie des Polaires réciproques; we may state it thus:

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"If round a fixed point, as vertex, two angles of constant magnitudes be made to turn, so that the points in which two of their sides respectively meet two given right lines are always in the same right line with a given point, the right line joining the points in which the two other sides respectively meet the same two right lines will envelope a conic section which will touch those two right lines."

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NOTES AND ADDITIONS.

PAGE 5, § 6.—*If through the vertex of a cone of the second degree right lines be drawn perpendicular to its tangent planes, they will form another cone, which will be of the second degree.*

The analytical proof of this proposition, though not quite so short, may appear to some readers more satisfactory than the geometrical one.

Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

be the equation of a cone; the axes of coordinates being rectangular. Then the equation of a tangent plane at the point, (x', y', z') will be

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}-\frac{z'z}{c^2}=0.$$

Now, the perpendicular to this tangent plane, passing through the origin, makes angles with the axes of x, y, and z, whose cosines are respectively proportional to $\frac{x'}{a^2}$, $\frac{y'}{b^2}$, and $-\frac{z'}{c^2}$; but these cosines are also proportional to the coordinates X, Y, Z, of any point on that perpendicular. Hence we find $a^2X^2+b^2Y^2-c^2Z^2=0$, for the equation of the second or *supplementary* cone, which is of the second degree; and its tangent plane at the point (X, Y, Z) is evidently perpendicular to the side of the given cone passing through the point (x', y', z').

PAGE 9, § 17.—The theorem stated in this paragraph is a case of the following general one:

If two surfaces of the second order intersect along a plane curve, they will also intersect along a second plane curve, either real or imaginary.

In order to prove this, let us take the plane of the first curve of intersection as the plane of xy: then the equations of the two surfaces being,

$$Ax^2+A'y^2+A''z^2+Byz+B'xz+B''xy+Cx+C'y+C''z+1=0,$$

$$ax^2+a'y^2+a''z^2+byz+b'xz+b''xy+cx+c'y+c''z+1=0,$$

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we must have the coefficients of the terms, independent of x, respectively equal in both; since the traces of the two surfaces upon the plane of xy are identical; that is, we must have,

$$A = a,\ A' = a',\ B'' = b'',\ C = c,\ C' = c'.$$

Hence, subtracting and dividing by x, we get,

$$(A'' - a'')x + (B - b)y + (B' - b')x + C'' - c'' = 0,$$

which is the equation of a second plane, on which the surfaces intersect.

The normal, at any point on the surface of a cone of the second degree, meets the plane of its least section at the centre of the sphere, passing through the two subcontrary circular sections which intersect in the point at which the normal is drawn.

It is easy to see that the centre of any sphere passing through two subcontrary circular sections, must lie in the plane of the least section of the cone: for the planes of the circular sections are perpendicular to that plane, and their centres lie in it. But further, if the two circular sections meet at a point on the surface of the cone, the sphere passing through them must necessarily touch the cone, so that its centre must lie on the normal at that point. Thus, we have proved that the centre of the sphere is at the point where the normal meets the plane of the least section of the cone.

The preceding proposition leads to an easy and direct proof of the first theorem in No. 24, p. 13.

Let VA, VB, be the two sides of the cone contained in the plane of its least section; and let AB, A'B', be the traces on this plane of the two subcontrary circular sections which pass through a point P on the surface of the cone, if perpendiculars be drawn to AB, A'B', at their middle points M, M', they will meet at O, the centre of the sphere passing through the two subcontrary sections.

Now, the plane touching the cone at P will also touch the sphere, and consequently it will be perpendicular to the radius OP: therefore, the angles, which the tangent plane makes with the two planes of circular section, are respectively equal to the angles which OP makes with OM, OM'; or, what is the same thing, to the angles AOM, A'OM'. But either the sum or the difference of these two angles is constant, and equal to the supplement of the angle AVB. Hence we have proved that

The sum or the difference of the angles, which each tangent plane to a cone of the second degree makes with the two cyclic planes, is constant.

It is easy to extend to the surfaces of the second order in general, the construction just given for determining the centre of the sphere passing through two subcontrary circular sections that meet at a point on the surface.

Let

$$Mx^2 + M'y^2 + M''x^2 + Nz + N'y + N''x = 0 \qquad (1)$$

be the equation of the surfaces of the second order; the axes of

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coordinates being rectangular and parallel to conjugate diameters, and the origin on the surface.

In this equation, without at all diminishing its generality, we may assume that M is intermediate between M′ and M″, regard being had to the *signs* as well as to the actual magnitude of these coefficients: for this relation between M, M′, M″, may be always obtained by interchanging the names of the coordinates. Let

$$(z-z')^2+(y-y')^2+(x-x')^2-r^2=0 \qquad (2)$$

be the equation of a sphere, in which x', y', z', r, are indeterminate. Multiplying this latter equation by M, and subtracting from the former, we get,

$$\left.\begin{aligned}&(M'-M)y^2+(M''-M)x^2+(N+2Mz')z\\&+(N'+2My')y+(N''+2Mx')x-M(z'^2+y'^2+x'^2-r^2)\end{aligned}\right\}=0. \qquad (3)$$

If the indeterminate quantities in this equation, x', y', z', r, be so assumed that it shall represent a system of two planes, these planes must intersect the given surface along two circles. Such a determination of x', y', z', r, may be effected in an infinite number of different ways; but in all cases we must have

$$N+2Mz'=0; \qquad (4)$$

as appears from comparing equation (3) with the product of the equations of two planes: and this shows that all the planes of the circular sections are perpendicular to the plane of xy.

The coefficient of z in equation (3) being thus made $=0$, the conditions to be fulfilled, in order that it shall represent two planes, are, 1st, that

$$\frac{(N'+2My')^2}{M'-M}+\frac{(N''+2Mx')^2}{M''-M}-4M(z'^2+y'^2+x'^2-r^2)=0; \qquad (5)$$

and 2nd, that the product $(M-M')(M-M'')$ may not be positive. This latter condition is already complied with, since M is intermediate between M′ and M″.

The value of z being determined by equation (4) we still have three quantities, x', y', r, to dispose of, and may at once satisfy equation (5) by putting

$$N'+2My'=0,\quad N''+2Mx'=0,\quad z'^2+y'^2+x'^2-r^2=0.$$

By these assumptions, along with equation (4), the equation (3) is reduced to the form,

$$(M'-M)y^2+(M''-M)x^2=0,$$

representing two planes perpendicular to the plane of xy, and whose traces on that plane make angles with the axis of x, the tangents of

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which are $+\sqrt{\frac{M''-M}{M-M'}}$, and $-\sqrt{\frac{M''-M}{M-M'}}$. These values of the tangents must plainly be real.

The values for the coordinates of the centre of the sphere, and for its radius, are next to be considered. The three equations,

$$N + 2Mx' = 0,\quad N' + 2My' = 0,\quad N'' + 2Mx' = 0,$$

show that the centre is on the normal to the given surface at the origin: for x', y', z' are proportional to N'', N', N; and the equation of the tangent plane at the origin is,

$$Nz + N'y + N''x = 0.$$

Moreover, the equation, $N + 2Mz' = 0$, shows that the centre of the sphere is in the principal plane of the surface, parallel to the plane of xy.

Hence, we have proved that in general,

The circular sections, passing through a given point on a surface of the second order, may be determined by the following construction: At the given point let a normal to the surface be drawn to meet the principal plane, to which the circular sections are perpendicular: the point of intersection will be the centre, and the normal itself will be the radius, of the sphere which cuts the given surface in the two circular sections passing through the given point.

In the analysis of this question there is one case which requires particular notice, viz. that of the hyperbolic paraboloid, where M' and M'' have different signs, and $M = 0$. Here we have real values for the tangents of the angles between the planes of circular section and the axis of x: but the values of x', y', z', become infinite, shewing that the sphere degenerates into a plane, and the circular into rectilinear sections.

Confined within the narrow limits of a note, I must either leave to the student, or defer until some other occasion, the fuller discussion of the analytical method which I have here used in determining the circular sections of the surfaces of the second degree. It will be found to conduct easily to many interesting results relative to these sections.

If we are only allowed to assume that the circular sections of any surface of the second degree are perpendicular to one of its principal planes, we may apply to these surfaces in general the geometrical proof given above in the case of the cone. For the sphere which passes through two circular sections that meet at a point on the surface, must touch the surface, and its centre must therefore be on the normal to the surface at that point. And since the two circular sections are perpendicular to a principal plane, the centre of the sphere passing through them must lie in that plane.

PAGE 12, § 22. These two theorems lead to the following properties of cones having the same cyclic planes or the same focal lines.

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When two cones of the second degree have a common vertex and the same cyclic planes, if a plane be drawn through their vertex cutting the two cones, the sides along which it meets one cone will respectively make equal angles with the sides of the other cone, through which the secant plane passes.

When two cones of the second degree have a common vertex and the same focal lines, the angle contained between two planes touching the two cones will be equal to that contained between the two other tangent planes which pass through the right line in which the two former tangent planes intersect.

Cones which have the same focal lines are called by M. Chasles *biconfocal cones:* in like manner, cones which have the same cyclic planes may be called *biconcyclic cones.* An investigation of the properties of these latter would be found to guide the student to many new theorems analogous to those relative to similar and similarly placed conic sections.

In future, for the sake of brevity, I shall dispense with the enunciation of those properties of cones of the second degree which admit of being stated as properties of spherical conics: the latter mode of statement being in general more concise, as well as more readily understood.

PAGE 14, § 25.—I have not succeeded in finding the elementary geometrical proofs, which M. Chasles has promised to give, of the two theorems contained in this paragraph. The following demonstration of the theorem in the first column appears to be as simple as could be desired.

Let APB, A′PB′ be two circular sections of the cone, passing through a point P on its surface; let AB, A′B′, their traces on the plane of the least section of the cone, intersect in O; and at P let tangents be drawn to the two circular sections, respectively meeting AB and A′B′ in T and T′. Then OP is parallel to the line in which the cyclic planes intersect, and the tangents PT, PT′, are parallel to the lines in which the tangent plane at P intersects the cyclic planes. But the angles OPT, OPT′, are respectively double the angles PAO, PA′O, whose tangents are $\frac{PO}{AO}$ and $\frac{PO}{A'O}$: and since $PO^2 = AO.OB$, the product of the tangents of the halves of OPT, OPT′, is equal to $\frac{OB}{OA'}$, which is constant: OB and OA′ being to each other as the sines of the angles which the side of the cone BA′ makes with the two cyclic planes.

PAGE 43, § 6, 2nd Mem.—In his *Histoire de la Geometrie,* p. 236, M. Chasles informs us, that M. Magnus was anticipated in the discovery of the first of these two theorems by Fuss, a Russian geometer, who, in discussing the curve which is the locus of the vertices of all the spherical triangles having the same base and sum of sides, discovered that this curve is the intersection of a sphere with a cone of the second degree, whose vertex is at the centre of the sphere.—(*Nova Acta,* tom. iii. A. D. 1787.) The formulæ employed by Fuss conducted him to the following result, which he calls

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"*maxime memorabilem*," viz. that if the sum of the sides be given equal to half the circumference of the sphere, the locus of the vertex will be an arc of a great circle, whatever be the base of the triangle. This is evident from the most elementary considerations of spherical geometry. M. Chasles observes (*Histoire de la Geometrie*, p. 239,) that M. Steiner was the first who proved that the base of a spherical triangle whose area and vertical angle are given, envelopes a spherical conic.

PAGE 46, § 11.—From the two theorems in this article we may deduce the two following:

If two spherical conics have the same cyclic arcs, every arc of a great circle touching the inner curve meets the outer one in two points which are equally distant from the point of contact.

If two spherical conics have the same foci, two arcs drawn from a point in the outer curve to touch the inner one make equal angles with the tangent to the outer conic at the point from which the tangents are drawn.

These last two theorems are only particular cases of the two following:

If two spherical conics have the same cyclic arcs, every arc of a great circle which cuts both of them intersects one curve in two points, which are equally distant from the points in which this arc meets the other curve.

If two spherical conics have the same foci, the angle contained between two arcs touching the two curves will be equal to the angle between the two other tangent arcs which may be drawn to the conics from the same point.

PAGE 47, § 14.—If M. Chasles had not restricted himself to the consideration of the hemisphere on which the conic is traced, (see page 45, § 10,) he might have stated the theorems in this article more elegantly.

For each arc touching the conic meets a cyclic arc in two points diametrically opposite; so that, *if the points of intersection be rightly chosen*, we might assert that

(1.) Two tangent arcs to a spherical conic intersect the two cyclic arcs in four points, which lie in the circumference of a small circle, whose centre is the pole of the great circle passing through the two points of contact of the tangent arcs.

The four vector arcs, drawn from the two foci of a spherical conic to any two points of the curve, will all touch the same small circle, whose centre is the point of concourse of the two arcs touching the curve at these two points.

This mode of stating the preceding theorems is valuable; for from them we are led to those contained in § 16, by the aid of the following propositions, which are very useful in spherical geometry, and may be easily proved:

(2.) If an arc of a great circle, passing through a fixed point on the surface of the sphere, intersect a given small circle in two points, the product of the tangents of the semi-arcs, lying between these two points and the fixed point, will be constant.

If from any point in a fixed arc of a great circle tangent arcs be drawn to a given small circle, the product of the tangents of the semi-angles, which these tangent arcs make with the fixed arc, will be constant.

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PAGE 47, § 15.—These theorems might be readily deduced from those in No. 11, by the following method:

Let O be the point of intersection of the cyclic arcs; then, since the portion of a tangent arc AB, intercepted between the two cyclic arcs, is bisected at the point of contact P, the two *consecutive* tangents AB, A′B′, must bisect one another in P; hence, the elementary spherical triangles APA′, BPB′, are equal, and the area of the whole triangle AOB, will remain invariable; the sum of its three angles is therefore constant: but the angle AOB is fixed, therefore, the sum of the angles ABO, BAO, must be constant. Having proved the first theorem thus, we may deduce the second from it by reference to the supplementary cone. Precisely in the same way we might prove the first of the two following theorems, and then derive the second from it by means of the supplementary cone.

If two spherical conics have the same cyclic arcs, any arc touching the inner curve, will cut off from the outer one a segment of a constant area.

If from any point in the outer of two biconfocal conics, two tangent arcs be drawn to the inner curve, the sum of these two arcs and of the concave part of the circumference of the conic included between them will be constant.

PAGE 47, § 17.—By means of the known formula

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s-a)}{\sin b \sin c},$$

expressing the cosine of the half of one angle of a spherical triangle in terms of the three sides, we might at once show that if the base a and the sum of the sides b, c, of a spherical triangle be constant, the product of the sines of the two arcs drawn from the extremities of the base perpendicular to the arc bisecting the supplement of the vertical angle is constant, and equal to

$$\sin^2 \tfrac{1}{2}(b+c) - \sin^2 \tfrac{1}{2} a.$$

PAGE 48, § 21.—It is evident that any arc, passing through a fixed point, and cutting a spherical conic and the polar arc of the fixed point, will be harmonically divided.

PAGE 50, § 28.—If a plane quadrilateral be circumscribed about a circle, the angles which two opposite sides subtend at the centre will be supplemental. Hence we derive the following theorems:

A spherical quadrilateral being circumscribed about a spherical conic, if arcs be drawn from the pole of one of its cyclic arcs through the four vertices of the quadrilateral, so as to meet that cyclic arc, the arcs drawn to two adjacent vertices of the quadrilateral will include between them a portion of the cyclic arc supplemental to that included between the two remaining arcs.

A spherical quadrilateral being inscribed in a spherical conic, if arcs be drawn from one focus to the four points in which the four sides meet the corresponding director arc, the portion of that arc included between two adjacent sides of the quadrilateral will subtend at the focus an angle supplemental to that subtended by the portion of the director arc included between the two other sides.

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PAGE 50, § 29.—From the theorems in this article we may deduce two others, which are most powerful instruments in discussing questions of spherical geometry.

It is easy to show that if four arcs diverging from the same point o be cut in the points A, B, C, D, by any fifth arc, we shall always have

$$\frac{\sin AB \sin CD}{\sin AD \sin BC} = \frac{\sin AOB \sin COD}{\sin AOD \sin BOC};$$

or, as M. Chasles designates it, the *anharmonic relation* of the four points is the same as that of the four arcs.

Hence we arrive at the following theorems:

(1.) If from four fixed points on a spherical conic arcs be drawn to any fifth point on the curve, their anharmonic relation will be constant.

If four fixed tangent arcs be drawn to a spherical conic, any fifth tangent arc will cut them in four points, the anharmonic relation of which will be constant.

The first of these theorems, or rather its converse, may often be used with success when we have to investigate the locus described by the vertex of a spherical triangle, two of whose sides pass through two fixed points P, P', whilst it also fulfils some other conditions. Let C, C', C'', C''', be four positions of the vertex; then, if we can show that the anharmonic relation of the arcs PC, PC', PC'', PC''', is the same as that of the arcs P'C, P'C', P'C'', P'C''', it follows that the locus of C will be a spherical conic passing through the points P, P'.

And in like manner, the second theorem may be employed as advantageously in ascertaining the envelope of the base of a spherical triangle, two of whose angles A, B, are on given arcs L, L', and which is further limited by some other conditions. Let A, A', A'', A''', be four positions of one angle, and B, B', B'', B''', the four corresponding positions of the other, then, if we can show that the anharmonic relation of A, A', A'', A''', is the same as that of B, B', B'', B''', the six arcs AB, A'B', A''B'', A'''B''', L, L', must all touch the same spherical conic, which will be the required envelope. In what follows we shall give several examples of the application of these principles.

Six points, A, A', B, B', C, C', lying in the arc of a great circle, and corresponding to each other two by two, are said to be *in involution*, when the anharmonic relation of four of them is the same as that of their four conjugates:

For instance, if the anharmonic relation of A, B, C, C', is the same as that of A', B', C', C, the three couples of points A, A', B, B', C, C', are in involution. And it may be proved that if this relation holds for one set of four points and their conjugates, it will also hold for any other: that is, if the anharmonic relation of A, B, C, C', is the same as that of A', B', C', C, we shall also have the anharmonic relation of A, B, C, B', the same as that of A' B', C', B, and so on.

Again, from the definition which has been given for the involution of six points, it may be shown that if there be three or more couples of points such, that the first two couples, taken along with any other

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couple, form a system in involution, any three of these couples will likewise form a sytem in involution.

Three couples of arcs which pass through the same point are said to be in involution, if the anharmonic relation of any four of them is the same as that of their four conjugates: and such a system possesses properties analagous to those of a system of six points in involution.

Using the preceding definitions and properties of systems of points or arcs in involution, we may deduce the following theorems:

(2.) A spherical quadrilateral being inscribed in a conic, any transversal arc will cut the curve and the four sides of the quadrilateral in six points, which are in involution.

A spherical quadrilateral being circumscribed about a conic, the four arcs drawn to its four vertices from any point without the curve, taken along with the two tangent arcs drawn from the same point, form a system of six arcs in involution.

The theorem in the first column may be thus proved. Let the transversal arc meet two opposite sides of the quadrilateral, *ad*, *bc*, in A, A′, the two other sides *ab*, *cd*, in B, B′, and the conic in C, C′; then, by the first of the theorems (1), the anharmonic relation of the four arcs drawn from *a* to *d*, C, *b*, C′, is the same as that of the arcs drawn from *c* to *b*, C′, *d*, C; therefore the anharmonic relation of the points A, C, B, C′, is the same at that of A′, C′, B′, C; hence the points A, A′, B, B′, C, C′, are in involution.

From the preceding we may successively deduce the following pairs of theorems:

(3.) Two spherical conics being circumscribed about a spherical quadrilateral, any transversal arc of a great circle meets the two curves, and also two opposite sides of the quadrilateral in six points, which are in involution.

Two spherical conics being inscribed in a quadrilateral, four tangent arcs drawn to them from the same point, together with the two arcs drawn from the same point to two opposite vertices of the quadrilateral, form a system of six arcs in involution.

(4.) Three spherical conics being circumscribed about the same quadrilateral, any transversal arc of a great circle meets the three curves in six points, which are in involution.

Three spherical conics being inscribed in the same quadrilateral, the six tangent arcs, drawn to them from any point, will form a system of six arcs in involution.

PAGE 51, § 32.—To the theorems contained in this section we may be permitted to add the following, which are derived from known properties of the circle.

(1) The angle which an arc of a circle subtends at the centre is double of that which it subtends at any point in the remaining part of the circumference. Hence,

If from two fixed points on a spherical conic arcs be drawn through any third point on the curve, they will include between them, on one of the

If any tangent arc be drawn to a spherical conic, intersecting two fixed tangent arcs in two points, and through these points arcs be drawn to one of the

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cyclic arcs, a portion which will be the half of that included between two arcs drawn from the two fixed points through the pole of the cyclic arc.

foci of the conic, they will contain between them an angle which will be the half of that contained between the two vector arcs drawn to the points of contact of the fixed tangent arcs.

(2.) The portion of a tangent to a circle intercepted between two fixed tangents, subtends a constant angle at the centre. Hence

If two fixed tangent arcs be drawn to a spherical conic, and any third tangent arc be drawn meeting them in two points, the arcs passing through these two points and through the pole of a cyclic arc, will intercept on that cyclic arc a portion of a constant length.

If from two fixed points in a spherical conic arcs be drawn to any third point on the curve, and produced to meet one of the director arcs, they will intercept between them, on that director arc, a portion which will subtend a constant angle at the corresponding focus.

(3.) If from any point in a given right line tangents be drawn to a given circle, they will make with the given right line angles the product of the trigonometric tangents of whose halves is constant. Hence,

If from any point in a fixed arc two arcs be drawn touching a given spherical conic, they will intersect either of its cyclic arcs in two points, such that the product of the trigonometric tangents of the halves of their distances from the point in which the fixed arc meets the given cyclic arc will be constant.

If through a fixed point on the surface of the sphere any arc be drawn meeting a spherical conic in two points, the arcs drawn from either focus to these two points will make, with the arc drawn from the same focus to the given point, two angles, the product of the trigonometric tangents of whose halves will be constant.

(4.) If any right line be drawn through a fixed point in the plane of a given circle, intersecting it in two points, the radii drawn to these points will make angles with the radius passing through the given point such that the product of the trigonometric tangents of their halves will be constant. Hence we derive the two following theorems:

If any arc be drawn through a fixed point, intersecting a given spherical conic in two points, the arcs drawn through these two points and through the fixed point from the pole of one of the cyclic arcs of the conic will meet that cyclic arc in three points, such that the product of the trigonometric tangents of the halves of the distances of the first two from the third will be constant.

If from any point in a fixed arc tangent arcs be drawn to a given spherical conic, and arcs be drawn from one of the foci to the points in which these tangent arcs meet the corresponding director arc, these vector arcs will make, with the vector arc drawn to the point in which the fixed arc meets the director arc, two angles, the product of the trigonometric tangents of whose halves will be constant.

(5.) A plane triangle being inscribed in a circle, if from any

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point P in the circumference right lines Pa, Pb, Pc, be drawn, respectively meeting the three sides BC, CA, AB, and making with them equal angles on the same side of the lines Pa, Pb, Pc, then the three points, a, b, c, will lie in the same straight line. Hence,

If the three sides, a, b, c, of a spherical triangle inscribed in a conic, be successively produced to meet one of the cyclic arcs in the points α, β, γ, and equal portions $\alpha\lambda$, $\beta\mu$, $\gamma\nu$, be measured on the cyclic arc in the same direction, arcs drawn from the points λ, μ, ν, through any point on the curve will meet the sides a, b, c, in three points lying in the arc of a great circle.

From the three vertices of a spherical triangle ABC circumscribed about a conic, if arcs be drawn to one of the foci F, and three other vector arcs Fl, Fm, Fn, be drawn so that the angles AFl, BFm, CFn, may be equal and lie at the same side of the arcs FA, FB, FC; these three arcs, Fl, Fm, Fn, will meet any tangent arc in three points l, m, n, such that the arcs Al, Bm, Cn, will pass through the same point.

PAGE 51, § 33.—Two right lines and a fixed point being given, if a constant angle be made to turn round this point as vertex, the right line joining the points in which its sides meet the two given right lines will envelope a conic section (p. 62, iii. 1). Hence,

If a variable spherical angle turn round a fixed point on the surface of a sphere so as to intercept between its sides a constant segment on a given arc, the arc joining the points in which its sides meet two other fixed arcs will envelope a spherical conic touching those two fixed arcs.

If a constant spherical angle turn round a given point as vertex, the arcs joining the points in which its sides meet a fixed arc with two other fixed points will intersect in a point, the locus of which will be a spherical conic passing through those two fixed points.

PAGE 53, § 38.—The theorems contained in this article might be stated more generally as follows:

If two arcs be made to turn round two fixed points on a spherical conic, so as to intersect in any third point of the curve, the arc joining the points in which they respectively meet two fixed arcs will envelope a spherical conic touching these two fixed arcs.

Two fixed tangent arcs being drawn to a spherical conic, any third tangent arc will intersect them in two points, and the arcs respectively drawn through these points and through two fixed points will intersect in a point the locus of which will be a spherical conic passing through the two fixed points.

The principles stated in page 78 furnish us with easy proofs of these theorems; that in the first column may be proved thus: Let P, P′, be the two fixed points; C, C′, C″, C‴, four other points on the curve; then, since the anharmonic function of the arcs PC, PC′, PC″, PC‴, is the same as that of P′C, P′C′, P′C″, P′C‴, the anharmonic function of the four points A, A′, A″, A‴, in which PC, PC′, PC″, PC‴, intersect one given arc, will be the same as that of the four points B, B′, B″, B‴, in which P′C, P′C′, P′C″, P′C‴, meet the other given arc; therefore, the arcs AB, A′B′, A″B″, A‴B‴, are tangents to a spherical conic touching the two given arcs.

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The theorem in the second column might be similarly proved, or we may infer it from the first by reference to the supplementary cone.

PAGE 53, § 39.—Since the arc which bisects any angle of a spherical triangle divides the opposite side into segments, the sines of which are proportional to the sines of the conterminous sides, we infer from the theorem in this article, that

If the base of a spherical triangle be given, and also the ratio of the sines of the two remaining sides, the locus of the vertex will be a spherical conic, whose cyclic arcs will be in the two planes perpendicular to the radii of the sphere drawn to the points which divide the given base internally and externally, so that the sines of the segments may be in the given ratio.

If the vertical angle of a spherical triangle be given, and also the ratio of the sines of the two remaining angles, the base will envelope a spherical conic, whose foci will be the extremities of the radii of the sphere perpendicular to the planes of the two arcs which divide the given vertical angle internally and externally, so that the sines of the segments may be in the given ratio.

PAGE 53, § 40.—The theorem in the second column shows that if the base of a spherical triangle be given, and also the product of the cosines of the two remaining sides, the locus of the vertex will be a spherical conic.

PAGE 53, § 42.—To the theorems given by M. Chasles in this section, we may add the following:

(1). The chord in a circle joining the extremities of two radii which contain between them a constant angle envelopes a circle concentric with the given one. From this property of the circle we derive the following theorems relative to spherical conics.

A spherical conic and one of its cyclic arcs being given, if round the pole of this cyclic arc, as vertex, a spherical angle of variable magnitude be made to turn, whose sides intercept between them on the cyclic arc a portion of a constant length, the arc joining the points in which the sides of the moveable angle meet the given conic will envelope a second conic.

The given cyclic arc will be a cyclic arc of the new conic; and this arc will have the same pole with relation to the two curves.

A spherical conic and one of its foci being given, if round that focus, as vertex, a constant spherical angle be made to turn, and from the points in which the sides of this angle meet the director arc corresponding to the given focus two tangent arcs be drawn to the given conic, their point of concourse will generate a second spherical conic.

The given focus will be a focus of the new conic; and the corresponding director arc will be the same for the two curves.

(2). If right lines be drawn from two fixed points in the circumference of a circle through the extremities of any diameter, they will intersect in a point the locus of which will be circle, which passes through the two fixed points, and whose centre is the pole of the right line joining them. Hence we derive the following theorems.

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A spherical conic and one of its cyclic arcs being given, if arcs be drawn from two fixed points on the curve to the extremities of any arc passing through the pole of that cyclic arc, and terminated by the curve, they will intersect in a point the locus of which will be a second spherical conic.

The given cyclic arc will be a cyclic arc of the new conic, and its pole, with relation to that curve, will be the same as the pole, with relation to the given conic, of the arc joining the two fixed points.

A spherical conic and one of its foci being given, if tangent arcs be drawn to the curve from any point in the corresponding director arc, the arc joining the points in which these tangent arcs meet two fixed tangent arcs will envelope a second spherical conic.

The given focus will be one of the foci of the new conic, and its corresponding director arc, for that curve, will be the arc joining the points of contact of the two fixed tangent arcs.

(3). The principles stated in page 78 furnish us with easy proofs of the following theorems:

In a spherical triangle, if the base and the difference of the base angles be given, the locus of the vertex will be a spherical conic, passing through the extremities of the given base.

In a spherical triangle, if the vertical angle and the difference of the sides containing it be given, the base will envelope a spherical conic touching the two sides which contain the given angle.

(4). If two tangents to a parabola intersect at a constant angle, the radii vectores drawn from the focus to the two points of contact will also contain between them a constant angle. But in any conic section the point of concourse of the tangents at the extremities of two focal radii vectores, which contain between them a constant angle, will generate a conic section, (see page 62, iii. 2). Hence we derive the following very general properties of spherical conics:

If two tangent arcs to a given spherical conic intercept between them a segment of a constant length on a fixed tangent arc to the curve, their point of concourse will generate a second spherical conic.

If the segment intercepted on the fixed tangent be a quadrant, the point of concourse of the two tangent arcs will move along an arc of a great circle.

If a constant spherical angle turn round a fixed point on a given conic, as vertex, the arc joining the points in which its sides meet the curve will envelope a second spherical conic.

If the constant spherical angle be a right angle, the arc which it subtends in the spherical conic will pass through a fixed point.

(5). From the theorem in the second column it appears that if a constant angle turn round a fixed point on a conic section as vertex, the chord which it subtends will envelope a second conic section. Hence we deduce the following theorems:

If a variable spherical angle turn round a fixed point on a spherical conic so that the segment intercepted between its sides on a fixed arc may be of a constant length, the arc joining the

If a constant spherical angle turn round a fixed point as vertex, and if from the points in which its sides meet a fixed tangent arc to a given spherical conic two arcs be drawn touching the

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points in which these sides meet the conic will envelope a second spherical conic.

curve, their point of concourse will generate a second spherical conic.

(6). Again, from the second of these latter theorems we learn, that if a constant angle turn round a fixed point in the plane of a conic section as vertex, and if from the points in which its sides meet a fixed tangent two other tangents be drawn to the curve, their point of concourse will generate a second conic section. From this we deduce the following theorems:

If a variable spherical angle turn round a fixed point on the surface of a sphere so that the segment intercepted between its sides on a fixed arc may be of a given length, and if, from the points in which its sides meet a fixed tangent arc to a given spherical conic, two other tangent arcs be drawn to the curve, their point of concourse will generate a second spherical conic.

If a constant spherical angle turn round a fixed point on the surface of a sphere, and, from the points in which its sides meet a fixed arc, two arcs be drawn to a fixed point on a given spherical conic, the arc joining the points in which these two arcs meet the curve will envelope a second spherical conic.

As before, we might, from the theorem in the second column, deduce a property of the plane conic sections, and from it in turn derive a pair of theorems relating to spherical conics. In fact there is no limit to the number of theorems which might be obtained in this way.

Page 58, § 56.—The following theorems, which are more general than those given by M. Chasles in this article, may be proved by means of the principles laid down in page 78.

If any two spherical angles, each of invariable magnitude, turn round two fixed points as vertices, so that two of their sides intersect on a given spherical conic passing through the two fixed points, the point of intersection of their two other sides will generate a spherical conic, which will pass through the two fixed vertices of the moveable angles.

If along two fixed arcs any two segments, each of invariable magnitude, be made to move, so that the arc joining two of their extremities may be a tangent to a given spherical conic which touches the two fixed arcs, the arc joining their two other extremities will envelope a spherical conic which will touch the two fixed arcs along which the segments move.

The first theorem may be proved in the following manner:

Let P, P′, be the two fixed points round which the constant angles, MPC, MP′C, turn; let M, M′, M″, M‴, be four points on the given conic, and C, C′, C″, C‴, four positions of the point the locus of which is sought. Then, since the angles MPC, M′PC′, M″PC″, M‴PC‴, are equal, the anharmonic relation of the arcs PM, PM′, PM″, PM‴, is the same as that of the four arcs, PC, PC′, PC″, PC‴; and since the angles MP′C, M′P′C′, M″P′C″, M‴P′C‴, are equal, the anharmonic relation of the four arcs, P′M, P′M′, P′M″, P′M‴, is the same as that of the four arcs, P′C, P′C′, P′C″, P′C‴. But by the first

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theorem given in the note to page 50, § 29, the anharmonic relation of PM, PM′, PM″, PM‴, is the same as that of P′M, P′M′, P′M″, P′M‴; hence the anharmonic relations of the two systems of arcs PC, PC′, PC″, PC‴, and P′C, P′C′, P′C″, P′C‴, are the same: consequently a conic section will pass through the points P, P′, C, C′, C″, C‴.

The second theorem might be proved in a similar manner, or we may derive it from the first by means of the supplementary cone.

PAGE 66, cap. viii.—The theorems contained in this chapter are deduced by M. Chasles from preceding ones by means of that particular polar transformation in which a parabola is used as the auxiliary conic.—(See *Quetelet's Correspondance Mathematique et Physique*, Tom. V. p. 281.)

We may, however, arrive at them without having recourse to this method.

The first follows immediately from the theorem given in the note to page 51, § 33.

The second and third are particular cases of the theorems relative to the plane conic sections which may be derived from the theorems (5) and (6) first column, in the note to page 53, § 42.

PAGE 68, cap. ix. § 5.—This theorem, which M. Chasles obtains by a polar transformation, follows directly from that given in page 63, cap. iv. § 6.

PAGE 69, cap. x. § 5.—The theorem which M. Chasles here gives for the construction of the conic sections by points admits of a great extension.

If two spherical angles, which intercept segments of constant lengths on two fixed arcs, turn rourd two fixed points as vertices, so that two of their sides always intersect upon a given spherical conic passing through the two fixed points, the point of concourse of their two other sides will generate a second spherical conic passing through the two fixed points.

This may be readily proved. For let P, P′, be the two fixed points, and MPC, MP′C, the two constant angles: let M, M′, M″, M‴, be four points on the given conic, and C, C′, C″, C‴, four positions of the point whose locus is sought, then the anharmonic relations of the two systems of arcs, PM, PM′, PM″, PM‴, and PC, PC′, PC″, PC‴, must be the same; since the angles MPC, M′PC′, M″PC″, M‴PC‴, intercept equal segments on the same arc. For a similar reason, the anharmonic relations of the two systems P′M, P′M′, P′M″, P′M‴, and P′C, P′C′, P′C″, P′C‴, must be the same: but since the systems PM, PM′, PM″, PM‴, and P′M, P′M′, P′M″, P′M‴, have the same anharmonic relation, it follows that the two systems PC, PC′, PC″, PC‴, and P′C, P′C′, P′C″, P′C‴, must also have the same anharmonic relation: so that a spherical conic must pass through the six points P, P′, C, C′, C″, C‴.

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PAGE 70, cap. x. § 6.—This last theorem may also be extended.

If two spherical angles of constant magnitudes be made to turn round two fixed points, so that the arc joining the points in which two of their sides respectively meet two fixed arcs, may always be a tangent to a given spherical conic touching those two fixed arcs, the arc joining the points in which the two other sides respectively meet the same two arcs will envelope a second spherical conic, which will also touch these two fixed arcs.

For let P, P′, be the two fixed points, round which the constant angles APα, BP′β, turn; Aα, Bβ, being the two fixed arcs. Let A′, A″, A‴, be three other positions of the point A, and α′, α″, α‴, B′, B″, B‴, β′, β″, β‴, the corresponding positions of the points α, B, β; then the anharmonic relations of the two systems of points A, A′, A″, A‴, and α, α′, α″, α‴, are the same: likewise the anharmonic relations of B, B′, B″, B‴, and β, β′, β″, β‴, are the same: but if AB, A′B′, A″B″, A‴B‴, be tangents to the same conic which touches the two arcs Aα, and Bβ, the anharmonic relations of the two systems A, A′, A″, A‴, and B, B′, B″, B‴, will be the same, and therefore α, α′, α″, α‴, and β, β′, β″, β‴, will be similar systems; consequently the arcs αβ, α′β′, α″β″, α‴β‴, must all be tangents to aspherical conic which touches the two arcs Aα, Bβ.

The theorem just proved might have been deduced from the one contained in the preceding note, by employing two supplementary cones.

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APPENDIX.

ON THE APPLICATION OF ANALYSIS TO SPHERICAL GEOMETRY.

§ 1.—*On the Use of Spherical Coordinates.*

As the position of a point on a plane is determined by reference to two fixed right lines, or axes of coordinates, in that plane, so the position of a point on the surface of a sphere may be determined by referring it to two fixed arcs of great circles.

Through a point O on the surface of a sphere, which we shall call *the origin*, let two fixed arcs of great circles OX, OY, be drawn, and let the points X and Y be 90° distant from O; then if arcs be drawn from Y and X through any point P on the sphere, and respectively meeting OX and OY in M and N, the trigonometric tangents of the arcs OM, ON, are to be considered as the *spherical coordinates* of the point P, and we shall denote them by x and y. To the fixed arcs I propose to give the name of *arcs of reference.*

If the arcs OX, OY, XN, YM, be projected, by means of radii of the sphere, into right lines upon a tangent plane at the point O, the projections of the arcs XN, YM, will be respectively parallel to the projections of OX and OY; consequently, the projections of the arcs OM, ON, which are the trigonometric tangents of those arcs, or the spherical coordinates of the point P, are the rectilinear coordinates, in the tangent plane, of the projection of the point P; the axes of coordinates being the projections of the arcs of reference. This consideration leads to an important consequence, viz.: that *an equation of the n^{th} degree between the spherical coordinates x and y represents a curve formed by the intersection of the sphere with a cone of the n^{th} degree having its vertex at the centre of the sphere.*

Each side of such a cone would meet the surface of the sphere in two points diametrically opposite, which, however, will have the same spherical coordinates since $\tan\theta = \tan(180° + \theta)$.

Thus the equation of the first degree between x and y represents an arc of a great circle. An equation of the second degree represents a spherical conic, and so on.

In what follows, for the sake of simplicity, I shall suppose the arcs of reference to be at right angles to each other. In this case the arcs PM, PN, are perpendicular to OX and OY, and I call such a system of spherical coordinates *rectangular.*

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§ 2.—*The Equation of an Arc of a great Circle.*

The equation of the first degree between x and y may in general be written in the form

$$\alpha x + \beta y = 1. \qquad (1)$$

The great circle, represented by this equation, meets the arcs of reference in two points, the cotangents of whose distances from the origin are, α and β. Hence, denoting the coordinates of the pole of this great circle by x' and y', we shall have

$$x' = -\alpha \text{ and } y' = -\beta. \qquad (2)$$

It appears from this that if α and β, instead of being fixed, are merely connected by an equation of the first degree, the great circle (1) will turn round a fixed point. If the equation connecting α and β be of the second degree, the great circle will envelope a spherical conic. And, in general, if α and β be connected by an equation of the n^{th} degree, the great circle will envelope a spherical curve to which n tangent arcs may be drawn from a point without it. For the pole of this great circle will generate a curve formed by the intersection of the sphere with a cone of the n^{th} degree, and as a transversal arc of a great circle may meet this curve in n points, the great circle (1) may assume n different positions whilst it passes through the same point.

But the general equation of the first degree may be written in the form

$$y = mx + n \qquad (3)$$

and it is desirable to explain the geometric meaning of the constants m and n. As to n, it is evidently the coordinate of the point in which the great circle (3) meets the y arc of reference.

If we take another great circle whose equation is $y = mx$, and seek the coordinates of the point in which this great circle meets the circle (3), we should find infinite values for x and y. These two great circles will therefore intersect on the great circle of which the origin is the pole: and, consequently, in the equations of two arcs of great circles $y = mx + n$, $y = m'x + n'$, if $m = m'$, these two arcs will pass through the same point in the great circle of which the origin is the pole. Another geometric meaning may be given to the constant m: for α and β are evidently proportional to the cosines of the angles which the arc (1) makes with the arcs of reference, and $m = -\frac{\alpha}{\beta}$.

The equation of an arc of a great circle passing through a fixed point x', y', is

$$y - y' = m(x - x') \qquad (4)$$

where m is indeterminate; and the equation of the great circle passing through two given points x', y', and x'', y'', is

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$$\frac{y-y'}{x-x'}=\frac{y''-y'}{x''-x'} \tag{5}$$

§ 3. *To express the Distance between two Points on the Sphere in Terms of their Coordinates x', y', and x'', y''.*

Let δ be the distance between the points; ϱ' and ϱ'' their distances from the origin; and ω', ω'', the angles which ϱ' and ϱ'' make with the x arc of reference. Then we shall have the following equation

$$\cos\delta=\cos\varrho'\cos\varrho''+\sin\varrho'\sin\varrho''\cos(\omega''-\omega'),$$

but we also have

$$\begin{aligned}\cos\omega'&=x'\cot\varrho' & \cos\omega''&=x''\cot\varrho''\\ \sin\omega'&=y'\cot\varrho' & \sin\omega''&=y''\cot\varrho''\end{aligned}$$

and by squaring these last equations and adding them two by two we shall find

$$\cos\varrho'=\frac{1}{\pm\sqrt{1+x'^2+y'^2}}\text{ and }\cos\varrho''=\frac{1}{\pm\sqrt{1+x''^2+y''^2}}.$$

Substituting from the last six equations in the preceding one we obtain,

$$\cos\delta=\frac{1+x'x''+y'y''}{\pm\sqrt{(1+x'^2+y'^2)(1+x''^2+y''^2)}}. \tag{6}$$

From this we get the following formulæ which are more generally useful,

$$\sin\delta=\pm\sqrt{\frac{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2}{(1+x'^2+y'^2)(1+x''^2+y''^2)}} \tag{7}$$

$$\tan\delta=\frac{\pm\sqrt{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2}}{1+x'x''+y'y''}. \tag{8}$$

The last three formulæ have double signs, because they give the distance of the point x', y', from the point diametrically opposite to x'', y'', as well as from the point x'', y'' itself. And it must also be remembered that two points on the surface of a sphere may be joined by two different arcs of great circles, which taken together make up the whole circumference of a great circle.

§ 4.—*To express the Length of the Arc λ, drawn from a given Point x', y', at right Angles to the Arc of a great Circle whose Equation is $\alpha x+\beta y=1$.*

The required arc is the complement of the distance of the given point from the pole of the given circle, whose coordinates are $-\alpha$, and $-\beta$; substituting these values for x'', y'' in (6) we find,

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$$\sin \lambda = \frac{1 - \alpha x' - \alpha y'}{\pm \sqrt{(1 + \alpha^2 + \beta^2)(1 + x'^2 + y'^2)}}. \quad (9)$$

§ 5.—*To express the Angle θ between two Arcs whose Equations are $\alpha x + \beta y = 1$ and $\alpha' x + \beta' y = 1$.*

Since the angle between the arcs is equal to the distance between their poles, we find from (6)

$$\cos \theta = \frac{1 + \alpha\alpha' + \beta\beta'}{\pm \sqrt{(1 + \alpha^2 + \beta^2)(1 + \alpha'^2 + \beta'^2)}}. \quad (10)$$

This formula shows that the two arcs will be at right angles to each other if

$$1 + \alpha\alpha' + \beta\beta' = 0. \quad (11)$$

§ 6.—*To find the Equation of the great Circle passing through the Point x', y', and perpendicular to the Arc whose Equation is $\alpha x + \beta y = 1$.*

The great circle whose equation is required must pass through the pole of the given great circle, as well as through the point x', y'; hence from (5) it appears that its equation will be

$$\frac{y - y'}{x - x'} = \frac{\beta + y'}{\alpha + x'}. \quad (12)$$

§ 7.—*Transformation of spherical Coordinates.*

Spherical coordinates may be transformed by assuming a new origin and new arcs of reference. It is not, however, very easy to establish the general formulæ which enable us to pass from any one system of spherical coordinates to another. I have investigated these formulæ and will furnish them on another occasion, having at present no need to employ any but the simplest modes of transformation. In the first place, the formulæ which are to be used, in passing from one system of rectangular coordinates to another, whilst the origin remains the same, are

$$\begin{aligned} x &= x' \cos \alpha - y' \sin \alpha \\ y &= y' \cos \alpha + x' \sin \alpha \end{aligned} \quad (13)$$

α being the angle between the given and the new reference arcs of x. This becomes evident if we project the two systems of arcs of reference into right lines upon the tangent plane at the common origin. Next, we may change the origin, whilst the reference arc of x remains the same; the new system of coordinates as well as the given one being rectangular. The formulæ for this transformation are

$$x = \frac{x' + \tan \alpha}{1 - x' \tan \alpha} \quad y = \frac{y' \sec \alpha}{1 - x' \tan \alpha}. \quad (14)$$

where α denotes the distance between the two origins.

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To prove this, let OX, OY, be the given arcs of reference, and O'X, O'Y, the new ones; and from any point P draw arcs PM, PN, PN', respectively perpendicular to OX, OY, O'Y; then, as the arc PM passes through Y

$$\frac{\tan YN'}{\tan YN} = \frac{\cos PYN'}{\cos PYN} = \frac{\cos O'M}{\cos (OO' + O'M)};$$

hence, since cot YN, cot YN' are respectively equal to y and y' $OO' = \alpha$, and tan $O'M = x'$, we get

$$y = \frac{y' \sec \alpha}{1 - x' \tan \alpha};$$

also, since $OM = OO' + O'M$,

$$\tan OM = \frac{\tan OO' + \tan O'M}{1 - \tan OO' \tan O'M};$$

therefore

$$x = \frac{x' + \tan \alpha}{1 - x' \tan \alpha}$$

In like manner, if we wished to transfer the origin to a point on the y arc of reference, that arc being still retained, and the new system of coordinates being rectangular, we should have to use the formulæ,

$$x = \frac{x' \sec \beta}{1 - y' \tan \beta}, \quad y = \frac{y' + \tan \beta}{1 - y' \tan \beta}, \tag{15}$$

where β denotes the distance between the origins.

§ 8.—*A Curve on the Surface of the Sphere being represented by an Equation between spherical Coordinates, to determine the Equation of the great Circle touching it at a given Point x', y'.*

The equation of a great circle, passing through the points x', y', and x'', y'', on the curve, is

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'};$$

and if x'', y'' approach indefinitely near to x', y', this equation becomes

$$\frac{y - y'}{x - x'} = \frac{dy'}{dx'}; \tag{16}$$

This is the equation of the tangent arc at the point x', y': we may put it into the following symmetrical form:

$$\frac{x dy'}{x' dy' - y' dx'} + \frac{y dx'}{y' dx' - x' dy'} = 1. \tag{17}$$

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The formulæ (16) and (17) will evidently hold good whether the arcs of reference be rectangular or not.

§ 9.—*To determine the Equation of the normal Arc to a spherical Curve at a given Point x', y'.*

It appears from (12) that the equation of the arc passing through the point x', y', and perpendicular to the arc whose equation is (17) will be

$$(y-y')\,[dy'+x'\,(x'dy'-y'dx')]+(x-x')[dx'+y'\,(y'dx'-x'dy')]=0 \quad (18)$$

This is the equation of the normal arc at the point x', y'. The coordinates in this last formula are supposed to be rectangular.

§ 10.—The *spherical evolute* of a curve described on the surface of the sphere is the locus of the point in which two consecutive normal arcs to the curve intersect. In finding the spherical evolute, we may follow a course exactly analogous to that by which we investigate the evolutes of plane curves. We must take the equation of the normal arc (18) and differentiate it with respect to x' and y': supposing x and y to be constant; the resulting equation may be written in the form

$$\left.\begin{aligned}&(y-y')[d^2y'+x'\,(x'd^2y'-y'd^2x')+(x'dy'-y'dx')dx']\\&+(x-x')[d^2x'+y'\,(y'd^2x'-x'd^2y')+(y'dx'-x'dy')\,dy']\\&=dx'^2+dy'^2+(x'dy'-y'dx')^2\end{aligned}\right\}(19)$$

From this and the equation of the normal arc (18) we find

$$\left.\begin{aligned}y-y'&=\\&\frac{[dx'^2+dy'^2+(x'dy'-y'dx')^2]\,[dx'+y'\,(y'dx'-x'dy')]}{(dx'd^2y'-dy'd^2x')\,(1+x'^2+y'^2)+(x'dy'-y'dx')\,(dx'^2+dy'^2+(x'dy'-y'dx')^2)}\\x-x'&=\\&\frac{[dx'^2+dy'^2+(x'dy'-y'dx')^2]\,[dy'+x'\,(x'dy'-y'dx')]}{(dy'd^2x'-dx'd^2y')\,(1+x'^2+y'^2)+(y'dx'-x'dy')\,(dx'^2+dy'^2+(x'dy'-y'dx')^2)}\end{aligned}\right\}(20)$$

By these last equations taken along with that of the given curve and its first and second differentials, we may eliminate x', y', dx', dy', d^2x', d^2y'; and the resulting equation between x and y will be the required equation of the spherical evolute.

§ 11.—*To determine the Circle of the Sphere which osculates a spherical Curve at a given Point x', y'.*

The spherical coordinates of the centre of the osculating circle are evidently the x and y in formulæ (20). To ascertain the magnitude of this circle we must have recourse to the formula (8), from which we find the tangent of the arc joining the points x, y, and x', y'. In the formulæ (20) let us put for shortness,

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$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2 = A,$$

$$dx'd^2y' - dy'd^2x' = B,$$

$$B(1 + x'^2 + y'^2) + (x'dy' - y'dx')\,A = C,$$

then we shall find from them

$$y = \frac{B(1 + x'^2 + y')\,y' + A dx'}{C} \qquad x = \frac{B(1 + x'^2 + y'^2)\,x' - A dy'}{C} \tag{21}$$

and

$$yx' - xy' = \frac{A\,(x'dx' + y'dy')}{C}.$$

Hence,

$$C(1 + xx' + yy') = C - A(x'dy' - y'dx') + B(1 + x'^2 + y'^2)(x'^2 + y'^2)$$
$$= B(1 + x'^2 + y'^2)^2.$$

Also,

$$C^2[(x' - x)^2 + (y' - y)^2 + (xy' - yx')^2] = A^3(1 + x'^2 + y'^2).$$

Therefore, if γ be the arc joining the points x, y, and x', y',

$$\left.\begin{aligned} \tan\gamma &= \pm\frac{A^{\frac{3}{2}}}{B(1 + x'^2 + y'^2)^{\frac{3}{2}}} \\ &= \pm\frac{[dx'^2 + dy'^2 + (x'dy' - y'dx')^2]^{\frac{3}{2}}}{(1 + x'^2 + y'^2)^{\frac{3}{2}}\,(dx'd^2y' - dy'd^2x')}. \end{aligned}\right\} \tag{22}$$

This remarkable formula, exactly analogous to that which expresses the radius of the circle osculating a given plane curve at the point x', y', might be obtained from the equation of a circle of the sphere by the following method. It appears from formula (6) that the equation of a circle of the sphere, whose spherical radius is γ, and the coordinates of whose centre are x'', y'', is

$$\cos\gamma = \frac{1 + xx'' + yy''}{\sqrt{(1 + x''^2 + y''^2)\,(1 + x^2 + y^2)}},$$

now this equation may be written in the form

$$ax + by + c = \sqrt{1 + x^2 + y^2}, \tag{23}$$

where

$$a^2 + b^2 + c^2 = \sec^2\gamma,$$

and in order that the circle, whose equation is (23) may osculate a given spherical curve at the point x', y', we must be able to change x and y, into x' and y', both in the equation (23) and in its first and second differentials. Thus, a, b, c, are completely determined from the three equations

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$$ax' + by' + c = \sqrt{1 + x'^2 + y'^2},$$

$$adx' + bdy' = \frac{x'dx' + y'dy'}{\sqrt{1 + x'^2 + y'^2}}$$

$$ad^2x' + bd^2y' =$$
$$\frac{(x'd^2x' + y'd^2y')\,(1 + x'^2 + y'^2) + dx'^2 + dy'^2 + (x'dy' - y'dx')^2}{(1 + x'^2 + y'^2)^{\frac{3}{2}}}.$$

Hence, using the abbreviations employed in the preceding investigation, we obtain

$$a = \frac{\mathrm{B}\,(1 + x'^2 + y'^2)\,x' - \mathrm{A}dy'}{\mathrm{B}\,(1 + x'^2 + y'^2)^{\frac{3}{2}}}$$

$$b = \frac{\mathrm{B}\,(1 + x'^2 + y'^2)\,y' + \mathrm{A}dx'}{\mathrm{B}\,(1 + x'^2 + y'^2)^{\frac{3}{2}}},$$

$$c = \frac{\mathrm{B}\,(1 + x'^2 + y'^2) + \mathrm{A}\,(x'dy' - y'dx')}{\mathrm{B}\,(1 + x'^2 + y'^2)^{\frac{3}{2}}},$$

and these three equations give us, since

$$a^2 + b^2 + c^2 - 1 = \tan^2\gamma,$$

$$\tan^2\gamma = \frac{\mathrm{A}^2}{\mathrm{B}^2\,(1 + x'^2 + y'^2)^3}.$$

§ 12.—*To find the differential of the Arc of a spherical Curve.* The equation (7) may be put into the form

$$\sin\vartheta = \pm\sqrt{\frac{(x'' - x')^2 + (y'' - y')^2 + (x'\,(y'' - y') - y'\,(x'' - x'))^2}{(1 + x'^2 + y'^2)\,(1 + x''^2 + y''^2)}};$$

therefore, if we use ds to denote the differential of the arc of a spherical curve, we shall have

$$ds = \frac{\sqrt{dx'^2 + dy'^2 + (x'dy' - y'dx')^2}}{1 + x'^2 + y'^2}. \qquad (24)$$

This expression enables us to present the equation (22), which gives the tangent of the radius of the osculating circle, in a new form,

$$\tan\gamma = \pm\frac{ds^3\,(1 + x'^2 + y'^2)^{\frac{3}{2}}}{dx'd^2y' - dy'd^2x'}. \qquad (25)$$

§ 13.—*On the Use of polar Coordinates in spherical Geometry.* In the analytic geometry of the plane, polar coordinates may sometimes be employed more advantageously than rectilinear ones: so, it is sometimes most convenient to express the position of a point

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on the surface of the sphere by means of the vector arc SP drawn to it from a fixed point S, and the angle PSX between this vector arc and a fixed arc passing through S.

The following formulæ, in which ϱ denotes the arc SP, and ω the angle PSX, are those most commonly used in applications of the method of spherical polar coordinates.

A curve being represented by an equation between ϱ and ω, it may be required to find the angle θ between the vector arc SP and an arc of a great circle touching the curve at the point P. This may be readily done. For, let a vector arc be drawn to another point Q, on the curve infinitely near to P; and from Q let an arc QT be drawn perpendicular to SP; then

$$\tan \mathrm{QPT} = \frac{\mathrm{QT}}{\mathrm{PT}}.$$

But ultimately $\mathrm{QPT} = \theta$, $\mathrm{QT} = \sin\varrho\, d\omega$, and $\mathrm{PT} = d\varrho$, hence we obtain the desired formulæ

$$\tan\theta = \frac{\sin\varrho\, d\omega}{d\varrho}. \tag{26}$$

Since QP is ultimately the differential of the arc ds, we find

$$ds = \pm\sqrt{\sin^2\varrho\, d\omega^2 + d\varrho^2}. \tag{27}$$

and

$$\sin\theta = \frac{\sin\varrho\, d\omega}{ds}. \tag{28}$$

From (28) we obtain an expression for the arc p, drawn from S perpendicular to the tangent arc at P.

$$\sin p = \frac{\sin^2\varrho\, d\omega}{\pm\sqrt{\sin^2\varrho\, d\omega^2 + d\varrho^2}}. \tag{29}$$

Next, we may find the area of the elementary spherical triangle whose base is QP, the differential of the arc, and whose vertex is the pole S. The known formula

$$\tan\tfrac{1}{2}(\text{area}) = \frac{\tan\frac{1}{2}a\,\tan\frac{1}{2}b\,\sin\mathrm{C}}{1+\tan\frac{1}{2}a\,\tan\frac{1}{2}b\,\cos\mathrm{C}},$$

which gives the area of a spherical triangle in terms of two sides and the included angle, becomes, when we substitute ϱ for a and b, and $d\omega$ for c,

$$\text{area} = 2\sin^2\tfrac{1}{2}\varrho\, d\omega = (1-\cos\varrho)\, d\omega, \tag{30}$$

An elegant expression may be obtained for the radius of the osculating circle in terms of ϱ and p. Let C be the centre of this circle, let γ be its radius, and let the arc SC be denoted by ϑ: then, using the notation already established, we shall have

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$$\cos \partial = \cos \varrho \cos \gamma + \sin \varrho \sin \gamma \sin \theta,$$

or,

$$\cos \partial = \cos \varrho \cos \gamma + \sin p \sin \gamma,$$

and this equation may be differentiated on the supposition that ∂ and γ remain constant. Thus we obtain

$$\tan \gamma = \frac{\sin \varrho \, d\varrho}{\cos p \, dp}. \tag{31}$$

If we differentiate (29), considering ω as the independent variable, we shall find $\cos p \, dp$ in terms of ϱ and ω; so that we arrive finally at the following equation which gives the tangent of the radius of the osculating circle in terms of ϱ and ω,

$$\tan \gamma = \frac{\pm (\sin^2 \varrho \, d\omega^2 + d\varrho^2)^{\frac{3}{2}}}{(\sin^2 \varrho \cos \varrho \, d\omega^2 + 2 \cos \varrho \, d\varrho^2 - \sin \varrho \, d^2\varrho) \, d\omega}. \tag{32}$$

§ 14.—*Mode of passing from rectangular to polar spherical Coordinates.*

To effect this change we have only to put

$$x = \tan \varrho \cos \omega, \text{ and } y = \tan \varrho \sin \omega. \tag{33}$$

From these equations we may deduce the following formula which is often useful,

$$d\omega = \frac{x dy - y dx}{x^2 + y^2}. \tag{34}$$

§ 15.—*Quadrature of a spherical Curve given by an Equation between rectangular spherical Coordinates.*

We may derive from (34) and (30) a formula for the quadrature of a spherical curve, which is given by an equation between rectangular spherical coordinates.

$$\text{area} = \int \frac{x dy - y dx}{x^2 + y^2} - \int \frac{x dy - y dx}{(x^2 + y^2) \sqrt{1 + x^2 + y^2}}. \tag{35}$$

The portion of the sphere whose area is determined by this last formula is included by an arc of the curve and two arcs of great circles drawn to its extremities from the origin. But we may derive from (30) a formula for the quadrature of spherical curves more analagous to that which is ordinarily employed in the quadrature of plane curves.

Let PM, P′M′, be two arcs drawn perpendicular to OX from two consecutive points on the curve. It is to be observed that the letters here employed refer to the construction indicated in § 1. Then the area of the elementary spherical quadrilateral PMM′P′, which we shall consider as the differential of the area of the curve described by P, may be easily found. For, Y being the pole of OX, since the area of

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the elementary spherical triangle PYP′ is by (30) equal to

$$(1-\cos \text{PY})\ \text{MM}' = (1-\sin \text{PM})\ \text{MM}',$$

it appears that the area of

$$\text{PMM}'\text{P}' = \sin \text{PM}.\text{MM}'.$$

Now

$$\sin \text{PM} = \sin \varepsilon \sin \omega = \frac{y}{\sqrt{1+x^2+y^2}},$$

and

$$\text{MM}' = d(\tan)^{-1}x = \frac{dx}{1+x^2},$$

therefore,

$$\text{area} = \int \frac{ydx}{(1+x^2)\sqrt{1+x^2+y^2}}. \qquad (36)$$

§ 15.—*Equation of a spherical Conic in rectangular Coordinates.*

Any equation of the second degree between the spherical coordinates x and y, represents a spherical conic.—(See § 1.) But in order to obtain the equation of a spherical conic in a simple form, we may take its principal diametral arcs for the arcs of reference. The equation of the conic referred to them will be

$$a^2y^2 + b^2x^2 = a^2b^2, \qquad (37)$$

where a and b are the tangents of the distances of the origin from the points in which the curve meets the x and y arcs of reference.

After what has been said in § 1, it is scarcely necessary to give any detailed proof of this. For, if the conic and its diametral arcs be projected by means of radii of the sphere upon a plane touching the sphere at the centre of the conic, the projection of the curve will be an ellipse having for its semiaxes the lines a and b, which are the projections of the principal semidiametral arcs of the spherical conic. Moreover, the equation of the spherical conic is identically the same as that of the ellipse referred to its principal axes. It will be convenient to denote the greatest and least semidiametral arcs, or as I shall henceforth call them, semidiameters of the conic by α and β, so that $a = \tan \alpha$, $b = \tan \beta$. We may define the foci of the conic as two points on the greatest diameter, whose distances from the centre, $+\gamma$ and $-\gamma$, are determined by the equation

$$\cos \gamma = \frac{\cos \alpha}{\cos \beta}. \qquad (38)$$

From this relation we find

$$\frac{\sin^2\gamma}{\sin^2\alpha} = \frac{a^2-b^2}{a^2}, \quad \frac{\tan^2\gamma}{\tan^2\alpha} = \frac{a^2-b^2}{a^2(1+b^2)}, \text{ and } \frac{\sin^2 2\gamma}{\sin^2 2\alpha} = \frac{(a^2-b^2)(1+b^2)}{a^2}. \qquad (39)$$

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These three quantities

$$\frac{\sin\gamma}{\sin\alpha}, \frac{\tan\gamma}{\tan\alpha}, \text{ and } \frac{\sin 2\gamma}{\sin 2\alpha},$$

as we shall see presently, will be found to appear in theorems and formulæ relating to spherical conics, in place of the excentricity which we meet in the theory of the plane conic sections. I shall denote them respectively by e, ϵ, and ϵ'.

If in the formulæ already given for the transformation of coordinates (14) we make $\alpha = 90°$, they would become

$$x = -\frac{1}{x'}, \text{ and } y = -\frac{y'}{x'}.$$

Substituting these values of x and y in (37), and removing the accents from x' and y', we should obtain the equation

$$a^2 b^2 x^2 - a^2 y^2 = b^2,$$

which is that of a spherical conic; the y and x arcs of reference being the great circles respectively lying in the principal plane, and in the plane of the greatest section of the cone whose generatrices are the radii drawn to all the points of the conic.

Transferring the origin from the centre to the vertex at the extremity of the greatest diameter, by putting $-a$ for $\tan\alpha$, in formulæ (14), we should find for the equation of the conic

$$y^2 = \frac{2b^2}{a}x - \frac{b^2(1-a^2)}{a^2}x^2. \qquad (41)$$

If $a = 1$, that is, if the greatest diameter of the curve becomes a quadrant, the last equation loses the term involving x^2, and the curve becomes what Mr. Davies calls a spherical parabola.

The quantity $\frac{2b^2}{a}$ appears to hold the same place in the theory of the spherical conics that it does in that of the plane conic sections. It may be called the *principal parameter*, and denoted by p.

Transferring the origin from the centre to the focus, by putting $\pm\gamma$ for α in formulæ (14) and afterwards removing the accents from x' and y', we should obtain the equation of a spherical conic in the following form,

$$y^2 + x^2 = (\mp p \mp \epsilon' x)^2. \qquad (42)$$

Now, this equation represents, not only the spherical conic, but also the plane curve formed by the intersection of the tangent plane at the focus with the cone whose generatrices are the radii of the sphere drawn to all the points of the conic: and it is evident from the form of equation (42) that this plane curve will be a conic section having its focus at the point of contact, its semiparameter equal to p, and its excentricity equal to ϵ'.

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Making $x = 0$ in (42), we shall find that the tangent of the arc, drawn from the focus to the curve, and perpendicular to the greatest diameter, is equal to half the principal parameter.

Again, without resorting to the formula (22), we may show that the tangent of the radius of the small circle osculating the conic at the extremity of the greatest diameter, is half the principal parameter. In order to prove this, I must premise the following proposition.

If an arc k be drawn from any point in the circumference of a small circle of the sphere, perpendicular to a fixed diametral arc d of the circle, and dividing it into two segments s and s', we should always have

$$\tan^2 k = \frac{\sin s \sin s'}{\cos^2 \frac{1}{2} d} :$$

and the value of $\frac{\tan^2 k}{2 \sin s'}$, when k and s' are both $= 0$, is evidently $\tan \frac{1}{2} d$.

Hence, if the equation of a spherical curve be given in rectangular coordinates, the arcs of reference being the tangent and normal arcs at a point on the curve, we should obtain the tangent of the radius of the circle of the sphere which osculates the curve at the origin by finding the value of $\frac{y^2}{2x}$, when x and y are each $= 0$.

Applying this principle to the spherical conic, represented by the equation (41), we should find the tangent of the radius of the circle osculating the conic at the origin to be equal to half the principal parameter.

§ 16.—*Polar Equation of a spherical Conic.*

The polar equation of a spherical conic referred to its centre as pole may be obtained from (37) by the method stated in § 14. We thus find it to be

$$\tan^2 \varrho = \frac{\tan^2 \beta}{1 - e^2 \cos^2 \omega}, \tag{43}$$

which also gives

$$\sin^2 \varrho = \frac{\sin^2 \beta}{1 - e^2 \cos^2 \omega}. \tag{44}$$

It appears from (43) that the sum of the squares of the cotangents of two rectangular semidiameters is constant.

Equation (44) shows that the orthographic projection of a spherical conic, upon a plane touching the sphere at the centre of the conic will be an ellipse.

Hence, if arcs PM, PN, be drawn from a point on a spherical conic perpendicular to its greatest and least diameters, we should have

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$$\frac{\sin^2 PN}{\sin^2 \alpha} + \frac{\sin^2 PM}{\sin^2 \beta} = 1. \qquad (45)$$

It would not, however, be advantageous in general to take the sines of the arcs drawn from a point on the sphere perpendicular to two fixed rectangular arcs as the spherical coordinates of that point.

Using these coordinates, and calling them x and y, we should find that the equation of a great circle was of the second degree, viz.:

$$\frac{x^2}{\sin^2 \alpha} + \frac{2xy}{\tan \alpha \tan \beta} + \frac{y^2}{\sin^2 \beta} = 1,$$

where α and β denote the distances of the origin from the points in which the great circle meets the arcs of reference.

The polar equation of a spherical conic referred to its focus as pole is found from (42) to be

$$\tan \varrho = \frac{\frac{1}{2} p}{1 \pm \varepsilon' \cos \omega}. \qquad (46)$$

We infer from this, that if any arc be drawn through the focus of a spherical conic, meeting the curve in two points, the sum of the trigonometric cotangents of the arcs lying between the focus and these two points will be constant.

It is easy to show that

$$\frac{\tan^2 \beta}{\tan \alpha} = \frac{\cos 2\gamma - \cos 2\alpha}{\sin 2\alpha}.$$

Equation (46) may therefore be put into the following form,

$$\tan \varrho = \frac{\cos 2\gamma - \cos 2\alpha}{\sin 2\alpha \pm \sin 2\gamma \cos \omega}. \qquad (47)$$

If we investigate by polar coordinates the locus of a point on the surface of a sphere the sines of whose distances from a fixed point on the sphere and a fixed arc are always in the ratio of m to 1, we should find for the equation of the locus

$$\tan \varrho = \frac{m \sin \delta}{1 - m \cos \delta \cos \omega}, \qquad (48)$$

where δ is the distance of the fixed point from the fixed arc.

Comparing this equation with (46) we have

$$m \sin \delta = \frac{1}{2} p, \text{ and } m \cos \delta = \varepsilon',$$

so that

$$m^2 = \frac{1}{4} p^2 + \varepsilon'^2, \text{ and } \tan \delta = \frac{p}{2\varepsilon'} \qquad (49)$$

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§ 17.—*Equations of the cyclic Arcs of a spherical Conic.*

A spherical conic being formed by the intersection of the sphere with a cone of the second degree, having its vertex at the centre of the sphere, the *cyclic arcs* of this conic are the great circles whose planes are parallel to two subcontrary circular sections of the cone. Consequently, if a plane be drawn touching the sphere at the pole of one of these great circles, it will intersect the cone in a circle. Now, the equation of that circle in the tangent plane is the same as that of the spherical conic, if the point of contact be made the origin of rectangular coordinates in both cases. Further, we may assume that the poles of the cyclic arcs are on the least diameter of the conic. The question of finding the cyclic arcs is, therefore, reduced to this,—to transfer the origin of rectangular spherical coordinates from the centre of the conic to a point in its least diameter, that diameter being retained as the y arc of reference, and the new coordinates being rectangular, so that in the transformed equation x^2 and y^2 may have the same coefficient. In order to effect this we must use the formulæ (15) for the transformation of coordinates, putting ψ in them instead of β, so as to avoid ambiguity, and then determine ψ by the condition that the coefficients of x^2 and y^2 may be equal. We thus obtain

$$\tan^2\psi = \frac{a^2 - b^2}{b^2(1+a^2)}.$$

Hence, we find the equations of the cyclic arcs of a conic given by the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

to be

$$y = \frac{\pm b\sqrt{1+a^2}}{\sqrt{a^2-b^2}}. \qquad (50)$$

Denoting by φ the angle between one of the cyclic arcs and the greatest diameter, we get from this, since $\varphi + \psi = 90^0$,

$$\cos\varphi = \frac{\tan\gamma}{\tan\alpha}, \text{ and } \sin\varphi = \frac{\sin\beta}{\sin\alpha}. \qquad (51)$$

These equations show that if two spherical conics have the same cyclic arcs, the quantity ϵ, in their polar equations (44) referred to the centre as pole, will be the same for both: consequently, *the ratio of the sines of those semidiameters of two biconcyclic conics which make equal angles with the greatest diametral arc will be constant: and hence, if two biconcyclic conics be projected orthographically upon a plane touching the sphere at their common centre, the projections will be two similar, similarly placed, and concentric ellipses.*

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§ 18.—*Equation of the great Circle touching a spherical Conic at a given Point x', y'.*

The equation of the spherical conic referred to its principal diameters being

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

we find from (16) the equation of the tangent arc at the point x', y', to be

$$a^2 y' y + b^2 x' x = a^2 b^2. \tag{52}$$

Hence, also, we may prove, in the same way as the similar result is arrived at in the theory of the plane conic sections, that if tangent arcs be drawn to a conic from a point without it, x', y', the equation of the arc joining the points of contact will be

$$a^2 y' y + b^2 x' x = a^2 b^2. \tag{53}$$

Since x and y may be interchanged with x' and y', without altering the form of the equation (53), it must also represent the locus of the point of concourse of the two tangent arcs drawn to a conic at the points where any arc passing through a fixed point x', y', meets the curve.

The arc touching the conic at the point x', y', meets the principal diameters of the conic in points, the tangents of whose distances from the centre are $\frac{a^2}{x'}$ and $\frac{b^2}{y'}$: call these X and Y, and we shall have, since $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$X^2 b^2 + Y^2 a^2 = X^2 Y^2.$$

This shows that the spherical conic, the tangents of whose principal semidiameters are X and Y, will pass through the point whose coordinates are a and b.

§ 19.—*Conjugate Diameters of a spherical Conic.*

Conjugate diameters of a spherical conic are those each of which passes through the pole of the other with relation to the conic. The poles of both of them are evidently on the great circle whose pole is the centre of the conic.

From what has been said in § 1, with reference to the meaning of the constants in the equation of an arc of a great circle, it appears that the diameter whose equation is

$$a^2 y' y + b^2 x' x = 0, \tag{54}$$

is conjugate to the diameter drawn to the point x', y', and whose equation is therefore,

$$x' y - y' x = 0;$$

for it meets the great circle whose pole is the centre of the conic in

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the same point that the tangent arc at the point x', y', meets that same circle.

Any two conjugate diameters of a spherical conic will evidently make with the greatest diameter two angles the product of whose tangents is constant and equal to $-\frac{b^2}{a^2}$.

The diameter conjugate to that which passes through the point x', y', on the conic, will meet the curve in two points whose coordinates x'', y'', are found from the equations (54) and (37) to be

$$x'' = \pm \frac{a}{b} y', \text{ and } y'' = \pm \frac{b}{a} x'. \tag{55}$$

Let us denote the semidiameter drawn to the point x', y', by r', and the semiconjugate diameter by r'', then as $\tan^2 r'' = x''^2 + y''^2$, we shall have

$$\tan^2 r'' = \frac{a^4 y'^2 + b^4 x'^2}{a^2 b^2}. \tag{56}$$

The equation of the tangent arc at that extremity of the conjugate diameter, for which the coordinates are

$$x'' = -\frac{a}{b} y', \text{ and } y'' = \frac{b}{a} x', \text{ is } x'y - y'x = ab. \tag{57}$$

It may be observed that since

$$x'^2 + x''^2 = a^2, \text{ and } y'^2 + y''^2 = b^2,$$

the sum of the squares of the tangents of two conjugate semidiameters of a spherical conic is constant.

§ 20.—*To determine the lengths of the Arcs drawn from the Centre or Focus perpendicular to Tangent Arc at the Point x', y'.*

Let λ be the arc drawn from the centre, perpendicular to the tangent arc at the point x', y': by the aid of formulæ (9) and (52), we find

$$\left.\begin{aligned} \sin \lambda &= \frac{a^2 b^2}{\sqrt{a^4 b^4 + a^4 y'^2 + b^4 x'^2}}, \\ \text{and} \\ \tan \lambda &= \frac{a^2 b^2}{\sqrt{a^4 y'^2 + b^4 x'^2}}. \end{aligned}\right\} \tag{58}$$

The last formula compared with (56) shows that $\tan r'' \tan \lambda = ab$: that is

In a spherical conic, the product of the trigonometric tangents of the perpendicular let fall from the centre of the conic on a tangent arc, and of the semidiameter conjugate to that which passes through the point of contact, is constant.

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Let l be the arc drawn from the focus, whose coordinate is $+ \tan \gamma$, $(= a_\iota)$ perpendicular to the tangent arc at the point x', y'; and let l' be the arc drawn from the other focus, whose coordinate is $- a_\iota$, perpendicular to the same tangent arc. Then, from formulæ (9) and (52), we have

$$\sin l = \frac{ab^2(a - \iota x')}{\sqrt{(a^4 b^4 + a^4 y'^2 + b^4 x'^2)(1 + a^2 \iota^2)}}, \tag{59}$$

and

$$\sin l' = \frac{ab^2(a + \iota x')}{\sqrt{(a^4 b^4 + a^4 y'^2 + b^4 x'^2)(1 + a^2 \iota^2)}}. \tag{60}$$

Now, the quantity $a^4 b^4 + a^4 y'^2 + b^4 x'^2$ may be put into the form $a^2 b^2 (1 + b^2)(a^2 - \iota^2 x'^2)$: consequently,

$$\sin l \sin l' = \frac{b^2}{1 + a^2}. \tag{61}$$

Therefore, in any spherical conic, the product of the sines of the arcs drawn from the two foci perpendicular to any tangent arc is constant.

§ 21.—*A spherical Conic being represented by the Equation* $a^2 y^2 + b^2 x^2 = a^2 b^2$, *to express the Distances* ϱ', ϱ'', *of a Point* x', y', *on the Curve from the two Foci.*

To accomplish this we have only to put $\pm a_\iota$ and 0, in place of x'' and y'', in formula (8): we thus find

$$\tan^2 \varrho' = \frac{(x' - a_\iota)^2 + y'^2 (1 + a^2 \iota^2)}{(1 + a_\iota x')^2}. \tag{62}$$

Now the quantity $(x' - a_\iota)^2 + y'^2 (1 + a^2 \iota^2)$ may be put into the form $(a - \iota x')^2$; we have, therefore,

$$\tan \varrho' = \frac{a - \iota x'}{1 + a_\iota x'}; \tag{63}$$

and in like manner

$$\tan \varrho'' = \frac{a + \iota x'}{1 - a_\iota x'}. \tag{64}$$

From the form of these two last equations it is plain that

$$\varrho' = \alpha - (\tan)^{-1} \iota x', \text{ and } \varrho'' = \alpha + (\tan)^{1-} \iota x';$$

hence

$$\varrho' + \varrho'' = 2\alpha. \tag{65}$$

Therefore, the sum of the distances of any point on a spherical conic from the two foci is constant.

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It is to be observed, that the tangent of the distance of any point x', y', on a spherical conic from either focus is a rational function of x'.

Had we employed formula (7) instead of (8) we should have found the following values of $\sin\varrho'$ and $\sin\varrho''$, which we shall have occasion presently to refer to:

$$\sin\varrho' = \frac{a - \imath x'}{\sqrt{(1 + a^2\imath^2)(1 + x'^2 + y'^2)}}. \tag{66}$$

$$\sin\varrho'' = \frac{a + \imath x'}{\sqrt{(1 + a^2\imath^2)(1 + x'^2 + y'^2)}}. \tag{67}$$

Since

$$\frac{\sin l}{\sin\varrho'} = \frac{\sin l'}{\sin\varrho''},$$

if follows that the arcs drawn from the two foci to any point on the curve make equal angles with the tangent arc at that point.

Let θ denote the angle between either of these vector arcs and the tangent arc; then we shall have

$$\sin^2\theta = \frac{\sin l \sin l'}{\sin\varrho' \sin\varrho''}.$$

Hence

$$\sin^2\theta = \frac{a^2 b^4 (1 + x'^2 + y'^2)}{a^4 b^4 + a^4 y'^2 + b^4 x'^2}. \tag{68}$$

§ 22.—*Equation of the normal Arc of a spherical Conic.*

We find from equations (12) and (53) that the equation of the arc passing through the point x', y', and perpendicular to the arc which joins the points of contact of the two tangent arcs drawn to the conic from the point x', y', is

$$\frac{y - y'}{x - x'} = \frac{y' a^2 (1 + b^2)}{x' b^2 (1 + a^2)} = \frac{y' \sin^2\alpha}{x' \sin^2\beta}. \tag{69}$$

If the point x', y', be on the curve, this same equation becomes that of the normal arc at the point.

By making y and x successively $= 0$ in (69), we find the coordinates x'' and y'', of the points in which the arc represented by equation (69) meets the greatest and least diameters to be

$$x'' = \frac{a^2 - b^2}{a^2 (1 + b^2)} x', \text{ and } y'' = \frac{b^2 - a^2}{b^2 (1 + a^2)} y'. \tag{70}$$

But the arc which joins the points of contact of the two tangent arcs

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drawn to the conic from the point x', y', meets the greatest and least diameters in points whose coordinates x_1 and y_1, are given by the equations

$$x_1 = \frac{a^2}{x'}, \text{ and } y_1 = \frac{b^2}{y'}.$$

comparing these values with those of x'' and y'', given in (70), we have

$$\left.\begin{aligned} x'' x_1 &= \frac{a^2 - b^2}{1 + b^2} = \tan^2\gamma, \\ y'' y_1 &= \frac{b^2 - a^2}{1 + a^2} = -\sin^2\gamma. \end{aligned}\right\} \quad (71)$$

These last equations show, that if two tangent arcs be drawn from any point to a spherical conic, and another arc be drawn from the same point perpendicular to the arc which joins the points of contact, these two arcs, which are at right angles to each other, will meet either of the principal diameters of the conic in two points, the product of the coordinates of which is constant.

Hence, if two tangent arcs be drawn to a spherical conic, and if the arc joining the points of contact touch a second spherical conic which has the same foci as the first, the arc joining the point of concourse of the tangents to the first conic with the point of contact on the second curve, will be a normal to the latter.

As a particular case of the last theorem we may deduce the following:

Any arc passing through the focus of a spherical conic is perpendicular to the arc joining that focus with the point of concourse of the tangent arcs drawn to the conic at the two points in which the arc passing through the focus meets the curve.

It appears from (51), that equation (69) is the same for all spherical conics which have the same cyclic arcs. Hence, if tangent arcs be drawn to one of two biconcyclic conics from a point on the other, the arc drawn from that point perpendicular to the arc joining the points of contact on the first conic, will be a normal to the other curve.

Equations (70) show that if any number of spherical conics, which have the same cyclic arcs, be cut by an arc perpendicular to either of the principal diameters common to all the curves, the normals at the points in which this arc meets the several conics will all pass through the same point on that principal diameter.

In order to find the length of the normal arc ν, drawn from the point x', y', to meet the greatest diameter we must put in (8) $y'' = 0$, and $x'' = \varepsilon^2 x'$; (see 70): thus we obtain

$$\tan^2\nu = \frac{x'^2(1-\varepsilon^2)^2 + y'^2(1+\varepsilon^4 x'^2)}{(1+\varepsilon^2 x'^2)^2};$$

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to this expression we may give an elegant and symmetrical form ; for

$$1 - \epsilon^2 = \frac{b^2}{a^2}(1 + a^2\epsilon^2), \text{ and } y'^2 = \frac{b^2}{a^2}(a^2 \quad x'^2);$$

the numerator in the value of $\tan^2\nu$ may, therefore, be written thus,

$$\frac{b^2}{a^2}[x'^2(1-\epsilon^2)(1+a^2\epsilon^2) + (a^2 - x'^2)(1+\epsilon^4 x'^2)],$$

which is evidently equal to

$$\frac{b^2}{a^2}(a^2 - \epsilon^2 x'^2)(1 + \epsilon^2 x'^2).$$

Expunging the common factor $1 + \epsilon^2 x'^2$, and observing that

$$a^4 b^4 + a^4 y'^2 + b^4 x'^2 = a^2 b^2 (1 + b^2)(a^2 - \epsilon^2 x'^2),$$

and

$$1 + x'^2 + y'^2 = (1 + b^2)(1 + \epsilon^2 x'^2), \tag{72}$$

we obtain finally

$$\tan^2\nu = \frac{a^4 b^4 + a^4 y'^2 + b^4 x'^2}{a^4(1 + x'^2 + y'^2)}. \tag{73}$$

In like manner we find the length of the normal arc ν', drawn from the point x', y', to meet the least diameter.

$$\tan^2\nu' = \frac{a^4 b^4 + a^4 y'^2 + b^4 x'^2}{b^4(1 + x'^2 + y'^2)}. \tag{74}$$

It is evident that

$$\frac{\tan\nu}{\tan\nu'} = \frac{b^2}{a^2}.$$

From equations (68), (73), and (74) we have

$$\tan\nu \sin\theta = \tfrac{1}{2}p, \text{ and } \tan\nu' \sin\theta = a.$$

Hence, since the angle between the normal and either of the focal vector arcs drawn to the point x', y', is the complement of θ, we derive the following theorem.

From the points where the normal to a spherical conic meets its greatest and least diameters, if arcs be drawn perpendicular to either of the focal vector arcs passing through the point on the curve at which the normal is drawn, these arcs will cut off constant portions from the vector arc.

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Comparing equations (66) and (67) with (73) and (74) we find

$$\tan \nu \tan \nu' = \sin \varrho \sin \varrho' (1 + a^2). \tag{75}$$

Let us denote by ζ the angle between the normal and the arc drawn from the centre to the point x', y', then we shall have

$$\sin \lambda = \sin r' \cos \zeta, \text{ and } \sin \lambda \tan \nu = b^2 \cos r';$$

therefore,

$$\tan \nu \tan r' \cos \zeta = b^2. \tag{76}$$

Consequently, if we denote by ξ the portion cut off from the normal by a perpendicular let fall upon it from the centre, we obtain

$$\tan \nu \tan \xi = b^2. \tag{77}$$

§ 23.—*Equations of the director Arcs of a spherical Conic.*

The director arc of a spherical conic is the locus of the point of concourse of the two tangent arcs drawn to the conic at the points where any arc passing through the focus meets the curve.

The equations of the director arcs corresponding to the two foci are found from (53) to be

$$x = \pm \frac{a}{\varepsilon}. \tag{78}$$

In formula (9), let us make

$$\beta = 0, \text{ and } \alpha = \pm \frac{\varepsilon}{a},$$

and it will give for the value of the sine of the perpendicular $\varkappa$ let fall from the point x', y', on the director arc

$$\sin \varkappa = \frac{a \mp \varepsilon x'}{\sqrt{(a^2 + \varepsilon^2)(1 + x'^2 + y'^2)}}. \tag{79}$$

Comparing this equation with (66) and (67), we see that the sines of the distances of any point on the conic from a focus and the corresponding director arc are to each other in a constant ratio. If m be the exponent of this ratio,

$$m^2 = \frac{a^2 + \varepsilon^2}{1 + a^2 \varepsilon^2}. \tag{80}$$

§ 24.—*Equation of the spherical Conic supplementary to the one represented by the Equation*

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

The spherical curve *supplementary* to a given one is the locus of the poles of all the great circles which touch the given curve.

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The equation of the tangent arc at the point x', y', to the conic represented by the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, has been shown to be

$$a^2 y' y + b^2 x' x = a^2 b^2.$$

The coordinates of the pole of this circle x'' and y'', are by (2)

$$x'' = -\frac{x'}{a^2}, \text{ and } y'' = -\frac{y'}{c^2}. \tag{81}$$

Hence, putting the values of x' and y' found from these last equations in $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, and removing the accents from x'' and y'', as no longer necessary, we find the equation of the proposed locus to be

$$b^2 y^2 + a^2 x^2 = 1; \tag{82}$$

as the equation of the given conic may be written in the form

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1,$$

it is plainly supplementary to the conic represented by the equation (82).

Since $a > b$, the foci of the supplementary conic are on the least diameter of the given one. Let us denote the greatest and least semidiameters of the supplementary conic by α' and β', and the distance between its foci by $2\gamma'$; then we shall plainly have

$$\alpha' + \beta = \alpha + \beta' = 90^\circ,$$

so that

$$\cos \gamma' = \frac{\cos \alpha'}{\cos \beta'} = \frac{\sin \beta}{\sin \alpha} = \sin \varphi.$$

Therefore, the *foci* of the supplementary conic are the poles of the cyclic arcs of the given conic: and, since the curves are mutually supplementary, the foci of the given conic are the poles of the cyclic arcs of the supplementary one.

But we may prove the existence of a more general relation between the two curves in the following manner:

The equation of the locus of the point of concourse of tangents drawn to the given conic at the points in which any arc passing through the point x', y', meets the curve, being

$$a^2 y' y + b^2 x' x = a^2 b^2;$$

the coordinates of the pole of this great circle are

$$-\frac{x'}{a^2}, \text{ and } -\frac{y'}{b^2}.$$

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Again, the equation of the locus of the point of concourse of tangents drawn to the supplementary conic at the points in which any arc passing through the point whose coordinates are

$$-\frac{x'}{a^2}, \text{ and } -\frac{y'}{b^2},$$

meets the curve, will be

$$x'x + y'y = -1;$$

and the coordinates of the pole of this great circle are x' and y'.

Thus, we have proved that to a point and its polar arc, with relation to a given spherical conic, correspond an arc and its pole with relation to the supplementary conic. It appears from this that to a focus and its director arc in a given conic, correspond a cyclic arc in the supplementary conic, and its pole with relation to that curve.

The two supplementary conics being thus connected, we are able, when a theorem has been proved with relation to points and arcs belonging to a spherical conic, to deduce from it at once another theorem relating to the corresponding arcs and points belonging to the supplementary conic.

§ 25.—*Evolute and osculating Circle of a spherical Conic.*

By differentiating the equation (37) twice we should find

$$y'dx' - x'dy' = \frac{b^2\,dx'}{y'}.$$

$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2 = \frac{a^4b^4 + a^4y'^2 + b^4x'^2}{a^4y'^2}\,dx'^2. \quad (83)$$

$$dx' + y'(y'dx' - x'dy') = (1 + b^2)\,dx'.$$

$$dy' + x'(x'dy' - y'dx') = -\frac{b^2x'}{a^2y'}(1 + a^2)\,dx'.$$

$$dy'\,d^2x' - dx'\,d^2y' = \frac{b^4\,dx'^3}{a^2y'^3}. \quad (84)$$

Making these substitutions in formulæ (20), and observing that

$$a^4b^4 + a^4y'^2 + b^4x'^2 = a^2b^2(1 + b^2)(a^2 - \epsilon^2x'^2),$$

and

$$(1 + x'^2 + y'^2) = (1 + b^2)(1 + \epsilon^2x'^2).$$

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We obtain finally the coordinates of the centre of the osculating circle

$$x = \frac{a^2 - b^2}{a^4 (1 + b^2)} x'^3, \text{ and } y = \frac{b^2 - a^2}{b^4 (1 + a^2)} y'^3.$$

So that the equation of the evolute will be

$$\left(\frac{x}{a'}\right)^{\frac{2}{3}} + \left(\frac{y}{b'}\right)^{\frac{2}{3}} = 1,$$

where

$$a' = \frac{a^2 - b^2}{a (1 + b^2)}, \text{ and } b' = \frac{b^2 - a^2}{b (1 + a^2)}.$$

Substituting from (83) and (84) in formula (22), we get for the semidiameter of the osculating circle

$$\tan \gamma = \frac{(a^4 b^4 + a^4 y'^2 + b^4 x'^2)^{\frac{3}{2}}}{a^4 b^4 (1 + x'^2 + y'^2)^{\frac{3}{2}}} = \frac{\tan^3 \nu}{p^2}.$$

§ 26.—*Rectification and Quadrature of a spherical Conic.*

The formula already given (24) for the differential of the arc of a spherical curve becomes, when we put in their values (83) and (72) for

$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2,$$

and

$$1 + x'^2 + y'^2,$$

$$ds = \frac{bdx \sqrt{a^2 - \varepsilon^2 x^2}}{a \sqrt{1 + b^2} (1 + \varepsilon^2 x^2) y}.$$

Let us make $x = a \sin \varphi$, then, by the equation of the conic (37), $y = b \cos \varphi$; and the differential may be brought into the following form

$$ds = \frac{1}{a \sqrt{1 + b^2}} \left\{ \frac{1 + a^2}{1 + a^2 \varepsilon^2 \sin^2 \varphi} - 1 \right\} \frac{d\varphi}{\sqrt{1 - \varepsilon^2 \sin^2 \varphi}}.$$

The arc is, therefore, represented by means of two elliptic integrals of the first and third orders, having ε for their common modulus; the parameter of the latter being $a^2 \varepsilon^2$.

To find the area of the conic we may employ the formula (36). In it let us make

$$x = a \cos \theta, \text{ and } y = b \sin \theta,$$

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as the equation of the conic permits, then we shall have

$$d(\text{area}) = \frac{b}{a\sqrt{1+a^2}} \left\{ 1 - \frac{1}{1-\sin^2\alpha \sin^2\theta} \right\} \frac{d\theta}{\sqrt{1-\sin^2\gamma \sin^2\theta}}.$$

The area may, therefore, be made to depend upon two elliptic integrals of the first and third orders; the parameter of the latter being $-\sin^2\alpha$, and their common modulus $\sin\gamma$.

It may be observed, that the elliptic integral of the third order which presents itself in the quadrature of a spherical conic is of the *circular* and not of the *logarithmic* kind, since $\sin^2\alpha$ is intermediate between $\sin^2\gamma$ and 1.

THE END.

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